Spoon Feeding Continuity & Differentiability

My name is Subhashish Chattopadhyay. I have been teaching for IIT-JEE, Various International Exams (such as IMO [International Mathematics Olympiad], IPhO [International Physics Olympiad], IChO [International Chemistry Olympiad], IGCSE [IB], CBSE, I.Sc, Indian State Board exams such as WB-Board, Karnataka PU-II etc since 1989. As I write this book in 2016, it is my 25th year of teaching. I was a Visiting Professor to BARC Mankhurd, Chembur, Mumbai, Homi Bhabha Centre for Science Education (HBCSE) Physics Olympics camp BARC Campus.
The selection for National Camp (for Official Science Olympiads - Physics, Chemistry, Biology, Astronomy) happens in the following steps....

1) **NSEP** (National Standard Exam in Physics) and **NSEC** (National Standard Exam in Chemistry) held around 24th November. Approx 35,000 students appear for these exams every year. The exam fees is Rs 100 each. Since 1998 the IIT JEE toppers have been topping these exams and they get to know their rank / performance ahead of others.

2) **INPhO** (Indian National Physics Olympiad) and **INChO** (Indian National Chemistry Olympiad). Around 300 students in each subject are allowed to take these exams. Students coming from outside cities are paid fair from the Govt of India.

3) The Top 35 students of each subject are invited at HBCSE (Homi Bhabha Center for Science Education) Mankhurd, near Chembur, BARC, Mumbai. After a 2-3 weeks camp the top 5 are selected to represent India. The flight tickets and many other expenses are taken care by Govt of India.

Since last 50 years there has been no dearth of “Good Books“. Those who are interested in studies have been always doing well. This e-Book does not intend to replace any standard text book. These topics are very old and already standardized.
There are 3 kinds of Text Books

- The thin Books - Good students who want more details are not happy with these. Average students who need more examples are not happy with these. Most students who want to “Cram” quickly and pass somehow find the thin books “good” as they have to read less !

- The Thick Books - Most students do not like these, as they want to read as less as possible. Average students are “busy” with many other things and have no time to read all these.

- The Average sized Books - Good students do not get all details in any one book. Most bad students do not want to read books of “this much thickness“ also !!

We know there can be no shoe that’s fits in all.

Printed books are not e-Books! Can’t be downloaded and kept in hard-disc for reading “later” ……….

So if you read this book later, you will get all kinds of examples in a single place. This becomes a very good “Reference Material”. I sincerely wish that all find this “very useful”.

Students who do not practice lots of problems, do not do well. The rules of “doing well” had never changed .... Will never change !
After 2016 CBSE Mathematics exam, lots of students complained that the paper was tough!

CBSE assures remedial measures for tricky and tough Class XII Math paper

After several students claimed that the Central Board of Secondary Education (CBSE) Class XII board Mathematics examination paper was 'tricky' and tough, the board has issued a clarification on remedial measures which are likely to be taken before evaluation.

The CBSE says that feedback received from various stakeholders like students, subject teachers and examiners will be put before the committee of subject experts.
In 2015 also the same complain was there by many students.

New Delhi: A senior Congress member on Thursday raised the issue of the tough mathematics question paper in the ongoing CBSE board examinations and asked the government to consider the issue "seriously".
In March 2016, students of Karnataka PU-II also complained the same, regarding standard 12 (PU-II Mathematics Exam). Even though the Math Paper was identical to previous year, most students had not even solved the 2015 Question Paper.

These complains are not new. In fact since last 40 years, (since my childhood), I always see this; every year the same setback, same complain!

In this e-Book I am trying to solve this problem. Those students who practice can learn.

No one can help those who are not studying, or practicing.
A very polite request:

I wish these e-Books are read only by Boys and Men. Girls and Women, better read something else; learn from somewhere else.
Preface

We all know that in the species “Homo Sapiens”, males are bigger than females. The reasons are explained in standard 10, or 11 (high school) Biology texts. **This shapes or size, influences all of our culture.** Before we recall / understand the reasons once again, let us see some random examples of the influence.

Random - 1

If there is a Road rage, then who all fight? (generally?). Imagine two cars driven by adult drivers. Each car has a woman of similar age as that of the Man. The cars “touch” or “some issue happens”. Who all comes out and fights? Who all are most probable to drive the cars?

( Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win )

Random - 2

Heavy metal music artists are all Men. Metallica, Black Sabbath, Motley Crue, Megadeth, Motorhead, AC/DC, Deep Purple, Slayer, Guns & Roses, Led Zeppelin, Aerosmith … the list can be in thousands. All these are grown-up Boys, known as Men.

( Men strive for perfection. Men are eager to excel. Men work hard. Men want to win. )
Apart from Marie Curie, only one more woman got Nobel Prize in Physics. (Maria Goeppert Mayer - 1963). So, ... almost all are men.


The best Tabla Players are all Men.

History is all about, which Kings ruled. Kings, their men, and Soldiers went for wars. History is all about wars, fights, and killings by men.

Boys start fighting from school days. Girls do not fight like this.

( Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win. )
The highest award in Mathematics, the “Fields Medal” is around since decades. Till date only one woman could get that. (Maryam Mirzakhani - 2014). So, ... almost all are men.


Actor is a gender neutral word. Could the movie like “Top Gun” be made with Female actors? The best pilots, astronauts, Fighters are all Men.
In my childhood had seen a movie named “The Tower in Inferno“. In the movie when the tall tower is in fire, women were being saved first, as only one lift was working....

Many decades later another movie is made. A box office hit. “The Titanic“. In this also.... As the ship is sinking women are being saved. Men are disposable. Men may get their turn later...

Movies are not training programs. Movies do not teach people what to do, or not to do. Movies only reflect the prevalent culture. Men are disposable, is the culture in the society. Knowingly, unknowingly, the culture is depicted in Movies, Theaters, Stories, Poems, Rituals, etc. I or you can’t write a story, or make a movie in which after a minor car accident the Male passengers keep seating in the back seat, while the both the women drivers come out of the car and start fighting very bitterly on the road. There has been no story in this world, or no movie made, where after an accident or calamity, Men are being helped for safety first, and women are told to wait.
Random - 9

Artists generally follow the prevalent culture of the Society. In paintings, sculptures, stories, poems, movies, cartoon, Caricatures, knowingly / unknowingly, “the prevalent Reality “ is depicted. The opposite will not go well with people. If deliberately “the opposite “ is shown then it may only become a special art, considered as a special mockery.

Random - 10

Men go to “girl / woman’s house” to marry / win, and bring her to his home. That is a sort of winning her. When a boy gets a “Girl-Friend “, generally he and his friends consider that as an achievement. The boy who “got / won “ a girl-friend feels proud. His male friends feel, jealous, competitive and envious. Millions of stories have been written on these themes. Lakhs of movies show this. Boys / Men go for “bike race “, or say “Car Race “, where the winner “gets “ the most beautiful girl of the college.

( Men want to excel. Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win. )

Prithviraj Chauhan ‘ went ` to “pickup “ or “abduct “ or “win “ or “bring “ his love. There was a Hindi movie (hit) song ... “Pasand ho jaye, to ghar se utah laye “. It is not other way round. Girls do not go to Boy’s house or man’s house to marry. Nor the girls go in a gang to “pick-up “ the boy / man and bring him to their home / place / den.
Random - 11

Rich people; often are very hard working. Successful business men, establish their business (empire), amass lot of wealth, with lot of difficulty. Lots of sacrifice, lots of hard work, gets into this. Rich people’s wives had no contribution in this wealth creation. Women are smart, and successful upto the extent to choose the right/rich man to marry. So generally what happens in case of Divorces? Search the net on “most costly divorces” and you will know. The women;( who had no contribution at all, in setting up the business / empire ), often gets in Billions, or several Millions in divorce settlements.

Random - 12

A standardized test of Intelligence will never be possible. It never happened before, nor ever will happen in future; where the IQ test results will be acceptable by all. In the net there are thousands of charts which show that the intelligence scores of girls / women are lesser. Debates of Trillion words, does not improve performance of Girls.
I am not wasting a single second debating or discussing with anyone, on this. I am simply accepting ALL the results. IQ is only one of the variables which is required for success in life. Thousands of books have been written on “Networking Skills”, EQ (Emotional Quotient), Drive, Dedication, Focus, “Tenacity towards the end goal” … etc. In each criteria, and in all together, women (in general) do far worse than men. Bangalore is known as “….. capital of India”. [Fill in the blanks]. The blanks are generally filled as “Software Capital”, “IT Capital”, “Startup Capital”, etc. I am member in several startup eco-systems / groups. I have attended hundreds of meetings, regarding “technology startups”, or “idea startups”. These meetings have very few women. Starting up new companies are all “Men’s Game” / “Men’s business”. Only in Divorce settlements women will take their goodies, due to Biased laws. There is no dedication, towards wealth creation, by women.

Random - 13

Many men, as fathers, very unfortunately treat their daughters as “Princess”. Every “non-performing” woman / wife was “princess daughter” of some loving father. Pampering the girls, in name of “equal opportunity”, or “women empowerment”, have led to nothing.

"Please turn it down - Daddy is trying to do your homework."
There can be thousands of more such random examples, where “Bigger Shape / size” of males have influenced our culture, our Society. **Let us recall the reasons**, that we already learned in standard 10 - 11, Biology text Books. In humans, women have a long gestation period, and also spends many years (almost a decade) to grow, nourish, and stabilize the child. (Million years of habit) Due to survival instinct Males want to inseminate. Boys and Men fight for the “facility (of womb + care)” the girl / woman may provide. Bigger size for males, has a winning advantage. Whoever wins, gets the “woman / facility”. The male who is of “Bigger Size”, has an advantage to win…. Leading to Natural selection over millions of years. In general “Bigger Males”; the “fighting instinct” in men; have led to wars, and solving tough problems (Mathematics, Physics, Technology, startups of new businesses, Wealth creation, Unreasonable attempts to make things [such as planes], Hard work….)

**So let us see the IIT-JEE results of girls.** Statistics of several years show that there are around 17, (or less than 20) girls in top 1000 ranks, at all India level. Some people will yet not understand the performance, till it is said that… year after year we have around 980 boys in top 1000 ranks. Generally we see only 4 to 5 girls in top 500. In last 50 years not once any girl topped in IIT-JEE advanced. Forget about Single digit ranks, double digit ranks by girls have been extremely rare. It is all about “good boys”, “hard working”, “focused”, “Bel-esprit” boys.

In 2015, Only 2.6% of total candidates who qualified are girls (upto around 12,000 rank). While 20% of the Boys, amongst all candidates qualified. The Total number of students who appeared for the exam were around 1.4 million for IIT-JEE main. Subsequently 1.2 lakh (around 120 thousands) appeared for IIT-JEE advanced.

IIT-JEE results and analysis, of many years is given at [https://zookeepersblog.wordpress.com/iit-jee-iseet-main-and-advanced-results/](https://zookeepersblog.wordpress.com/iit-jee-iseet-main-and-advanced-results/)

In Bangalore it is rare to see a girl with rank better than 1000 in IIT-JEE advanced. We hardly see 6-7 boys with rank better than 1000. Hardly 2-3 boys get a rank better than 500.

See [http://skmclasses.weebly.com/everybody-knows-so-you-should-also-know.html](http://skmclasses.weebly.com/everybody-knows-so-you-should-also-know.html)

Thousands of people are exposing the heinous crimes that Motherly Women are doing, or Female Teachers are committing. See [https://www.facebook.com/WomenCriminals/](https://www.facebook.com/WomenCriminals/)
Some Random Examples must be known by all

It is extremely unfortunate that the “woman empowerment” has created. This is the kind of society and women we have now. I and many other sensible Men hate such women. Be away from such women, be aware of reality.

Mother Admits On Facebook to Sleeping with 15 Yr Old Son, They Have a Baby Together - Abwayztumtup

Sometimes it hard to believe w From Abwayztumtup

‘Sex with my son is incredible - we’re in love and we want a baby’

Ben Ford, who ditched his wife when he met his mother Kim West after 30 years, claims what the couple are doing isn’t incest!

mirror.co.uk

Woman sent to jail for the rest of her life after raping her four grandchildren is described as the ‘most evil person the judge has ever seen

Edwina Louis rape...

See More

Former Shelbyville ISD teacher who had sex with underage student gets 3 years in prison

After a two day break over the weekend, A Shelby County Jury was back in the courtroom looking to conclude the trial of a former Shelbyville ISD teacher who had...

KLVN.COM | BY CALEB BEAMEZ

Woman sent to jail for raping her four grandchildren

A Ohio grandmother has been sentenced to four consecutive life terms after being found guilty of the rape of her own grandchildren. Edwina Louis, 54, will spend the rest of her life behind bars.

MAIL.CO.UK
End violence against women.

North Carolina Grandma Eats Her Daughter’s New Born Baby After Smoking Bath Salts
Henderson, North Carolina - A North Carolina grandmother of 4 and recovering drug addict, is now in custody after she allegedly ate her daughter's newborn baby.

http://www.nbcnews.com/.../attractive-girl-gang-raped-men-alleyswa...

Attractive Girl Gang Raped Men Into Alleyways Where Female Body Builder Would Attack Them
A Mexican street gang made up entirely of women has been accused of raping their female victims to lure men into alleyways and then beating them up and...

LATEST.COM

28-Year-Old Texas Teacher Accused of Sending Nude Picture to 14-Year-Old Former Student
http://www.breitbart.com/.../youngstown-woman-convicted-of-raping-

End violence against women.

Youngstown woman convicted of raping 1 year old is back in jail
A Youngstown woman who went to prison for raping a 1-year-old boy fifteen years ago is in trouble with the law again.

FOX19.COM

Women are raping boys and young men
Rape advocacy has been hijacked and belittled into a political agenda controlled by radicalized activists. Tim Purcell takes a raw, honest and well-supported look into the manufactured rape culture and...

AVOIDFEMENON.COM | BY TIM PURCELL

Bronx Woman Convicted of Poisoning and Drowning Her Children
Lisette Baranega researched methods on the Internet before she killed her son and daughter in 2012.

NYTIMES.COM | BY MARC SANTORA
In several countries or rather in several regions of the world, family system has collapsed, due to bad nature and naughty acts of women. Particularly in Britain, and America, almost 50% people are alone, lonely, separated, divorced or failed marriages. In 2013, 48% children were born out of wedlock. It was projected that by 2016, more than 51% children will be born, to unmarried mothers. In these developed countries "paternity fraud" by women, are close to 20%. You can see several articles in the net, and in wikipedia etc. This means 1 out of 5 children are calling a wrong man as dad. The lonely, alone "mothers" are frustrated. They see the children as burden. Love in the Society in general is lost, long time ago. The types of "Mothers" and "Women" we have now ..........
This is the type of women we have in this world. These kind of women were also someone's daughter.

Mother Stabs Her Baby 90 Times With Scissors After He Eats Her While Breastfeeding Him!

Eight-month-old Nita Ike was discovered by his uncle in a pool of blood needed 100 stitches after the incident; he is now recovering in hospital. Reports say his...

**CBSE Math Survival Guide - Continuity & Differentiability by Prof. Subhashish Chattopadhyay SKMClasses Bangalore Useful for IIT-JEE, I.Sc. PU-II, Boards, IGCSE IB AP-Mathematics and other exams**

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HURT FEMINISM BY DOING NOTHING

- Don’t help women
- Don’t fix things for women
- Don’t support women’s issues
- Don’t come to women’s defense
- Don’t speak for women
- Don’t value women’s feelings
- Don’t portray women as victims
- Don’t protect women

WITHOUT WHITE KNIGHTS FEMINISM WOULD END TODAY

High Priority

- Rich women
- Women
- Rich Men
- Girls
- Boys
- Animals
- Prisoners
- Men
- Poor Men

Low Priority

- They can get away with murder.
- They get all the rights with no responsibility and shelters for homeless women.
- They get bailouts and short prison sentences.
- They get educational benefits but no violence against kids Act.
- They have some support but don’t have any education that fits boys.
- They have animal rights and PETA.
- They get conjugal visits and 3 squares and a roof.
- Paid slaves.
- Nothing.

Who pays the most Taxes? This is why MGTOW exist.

# MGTOW

Professor Subhashish Chattopadhyay
Spoon Feeding Series - Continuity & Differentiability

Question

Prove that the function \( f(x) = 5x - 3 \) is continuous at \( x = 0 \), at \( x = -3 \) and at \( x = 5 \)

Answer:

The given function is \( f(x) = 5x - 3 \)

At \( x = 0 \), \( f(0) = 5 \times 0 - 3 = 3 \)

\[ \lim_{{x \to 0}} f(x) = \lim_{{x \to 0}} (5x - 3) = 5 \times 0 - 3 = -3 \]

\[ \therefore \lim_{{x \to 0}} f(x) = f(0) \]

Therefore, \( f \) is continuous at \( x = 0 \)

At \( x = -3 \), \( f(-3) = 5 \times (-3) - 3 = -18 \)

\[ \lim_{{x \to -3}} f(x) = \lim_{{x \to -3}} (5x - 3) = 5 \times (-3) - 3 = -18 \]

\[ \therefore \lim_{{x \to -3}} f(x) = f(-3) \]

Therefore, \( f \) is continuous at \( x = -3 \)

At \( x = 5 \), \( f(5) = 5 \times 5 - 3 = 25 - 3 = 22 \)

\[ \lim_{{x \to 5}} f(x) = \lim_{{x \to 5}} (5x - 3) = 5 \times 5 - 3 = 22 \]

\[ \therefore \lim_{{x \to 5}} f(x) = f(5) \]

Therefore \( f(x) \) is continuous around \( x = 5 \)

Question
Examine the continuity of the function \( f(x) = 2x^2 - 1 \) at \( x = 3 \)

**Answer:**

The given function is \( f(x) = 2x^2 - 1 \)

At \( x = 3 \), \( f(x) = f(3) = 2(3)^2 - 1 = 17 \)

\[
\lim_{x \to 3} f(x) = \lim_{x \to 3} (2x^2 - 1) = 2(3)^2 - 1 = 17
\]

\( \therefore \lim_{x \to 3} f(x) = f(3) \)

Thus, \( f \) is continuous at \( x = 3 \)

**Question**

Examine the following functions for continuity.

(a) \( f(x) = x - 5 \)

(b) \( f(x) = \frac{1}{x - 5}, x \neq 5 \)

(c) \( f(x) = \frac{x^2 - 25}{x + 5}, x \neq 5 \)

(d) \( f(x) = |x - 5| \)

**Answer:**

(a) The given function is \( f(x) = x - 5 \)

It is evident that \( f \) is defined at every real number \( k \) and its value at \( k \) is \( k - 5 \).

It is also observed that, \( \lim_{x \to k} f(x) = \lim_{x \to k} (x - 5) = k - 5 = f(k) \)

\( \therefore \lim_{x \to k} f(x) = f(k) \)

Hence, \( f \) is continuous at every real number and therefore, it is a continuous function.

(b) The given function is \( f(x) = \frac{1}{x - 5}, x \neq 5 \)

For any real number \( k \neq 5 \), we obtain
\[ \lim_{x \to k} f(x) = \lim_{x \to k} \frac{1}{k - 5} = \frac{1}{k - 5} \]

Also, \( f(k) = \frac{1}{k - 5} \) (As \( k \neq 5 \))

\[ \therefore \lim_{x \to k} f(x) = f(k) \]

Hence, \( f \) is continuous at every point in the domain of \( f \) and therefore, it is a continuous function.

(c) The given function is \( f(x) = \frac{x^2 - 25}{x + 5} \), \( x \neq -5 \)

For any real number \( c = -5 \), we obtain

\[ \lim_{x \to c} f(x) = \lim_{x \to -5} \frac{x^2 - 25}{x + 5} = \lim_{x \to -5} \frac{(x + 5)(x - 5)}{x + 5} = \lim_{x \to -5} (x - 5) = (c - 5) \]

Also, \( f(c) = \frac{(c + 5)(c - 5)}{c + 5} = (c - 5) \) (as \( c \neq -5 \))

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Hence, \( f \) is continuous at every point in the domain of \( f \) and therefore, it is a continuous function.

(d) The given function is \( f(x) = |x - 5| = \begin{cases} 5 - x, & \text{if } x < 5 \\ x - 5, & \text{if } x \geq 5 \end{cases} \)

This function \( f \) is defined at all points of the real line.

Let \( c \) be a point on a real line. Then, \( c < 5 \) or \( c = 5 \) or \( c > 5 \)

Case I: \( c < 5 \)

Then, \( f(c) = 5 - c \)

\[ \lim_{x \to c} f(x) = \lim_{x \to c} (5 - x) = 5 - c \]

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all real numbers less than 5.

Case II: \( c = 5 \)

Then, \( f(c) = f(5) = (5 - 5) = 0 \)

\[ \lim_{x \to 5} f(x) = \lim_{x \to 5} (5 - x) = (5 - 5) = 0 \]

\[ \lim_{x \to 5} f(x) = \lim_{x \to 5} (x - 5) = 0 \]

\[ \therefore \lim_{x \to 5} f(x) = \lim_{x \to 5} f(x) = f(c) \]
Therefore, \( f \) is continuous at \( x = 5 \).

Case III: \( c > 5 \).

Then, \( f(c) = f(5) = c - 5 \)
\[
\lim_{x \to c} f(x) = \lim_{x \to c} (x - 5) = c - 5
\]
\[
\therefore \lim_{x \to c} f(x) = f(c)
\]

Question

Prove that the function \( f(x) = x^n \) is continuous at \( x = n \), where \( n \) is a positive integer.

Answer:

The given function is \( f(x) = x^n \)

It is evident that \( f(x) \) is defined at all positive integers, \( n \), and its value at \( n \) is \( n^n \).

Then, \( \lim_{x \to n} f(x) = \lim_{x \to n} (x^n) = n^n \)
\[
\therefore \lim_{x \to n} f(x) = f(n)
\]

Therefore, \( f \) is continuous at \( n \), where \( n \) is a positive integer.

Question
Is the function \( f \) defined by
\[
 f(x) = \begin{cases} 
 x, & \text{if } x \leq 1 \\
 5, & \text{if } x > 1 
\end{cases}
\]

continuous at \( x = 0 \)? At \( x = 1 \)? At \( x = 2 \)?

Answer:

The given function \( f(x) \) is
\[
 f(x) = \begin{cases} 
 x, & \text{if } x \leq 1 \\
 5, & \text{if } x > 1 
\end{cases}
\]

At \( x = 0 \),

It is evident that \( f \) is defined at 0 and its value at 0 is 0.

Then,
\[
\lim_{{x \to 0}} f(x) = \lim_{{x \to 0}} x = 0
\]

\[
\therefore \lim_{{x \to 0}} f(x) = f(0)
\]

Therefore, \( f \) is continuous at \( x = 0 \)

At \( x = 1 \),
$f$ is defined at 1 and its value at 1 is 1.

The left hand limit of $f$ at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$$

The right hand limit of $f$ at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (5) = 5$$

$$\therefore \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

Therefore, $f$ is not continuous at $x = 1$

At $x = 2$,

$f$ is defined at 2 and its value at 2 is 5.

Then,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (5) = 5$$

$$\therefore \lim_{x \to 2} f(x) = f(2)$$

Therefore, $f$ is continuous at $x = 2$

Question
Find all points of discontinuity of \( f \), where \( f \) is defined by
\[
 f(x) = \begin{cases} 
 2x+3, & \text{if } x \leq 2 \\
 2x-3, & \text{if } x > 2 
\end{cases}
\]

Answer:

The given function \( f \) is defined as:
\[
 f(x) = \begin{cases} 
 2x+3, & \text{if } x \leq 2 \\
 2x-3, & \text{if } x > 2 
\end{cases}
\]

It is evident that the given function \( f \) is defined at all the points of the real line. Let \( c \) be a point on the real line. Then, three cases arise.

(i) \( c < 2 \)

(ii) \( c > 2 \)

(iii) \( c = 2 \)

Case (i) \( c < 2 \)

Then, \( f(c) = 2c+3 \)

\[
 \lim_{x \to c} f(x) = \lim_{x \to c} (2x+3) = 2c+3
\]
\[ \lim_{x \to 2} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x \neq 2 \)

Case (ii) \( c > 2 \)

Then, \( f(c) = 2c - 3 \)

\[ \lim_{x \to c} f(x) = \lim_{x \to c} (2x - 3) = 2c - 3 \]

\[ \therefore \lim_{x \to c} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 2 \)

Case (iii) \( c = 2 \)

Then, the left hand limit of \( f \) at \( x = 2 \) is,

\[ \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 3) = 2 \times 2 + 3 = 7 \]

The right hand limit of \( f \) at \( x = 2 \) is,

\[ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 3) = 2 \times 2 - 3 = 1 \]

It is observed that the left and right hand limit of \( f \) at \( x = 2 \) do not coincide.

Therefore, \( f \) is not continuous at \( x = 2 \)

Hence, \( x = 2 \) is the only point of discontinuity of \( f \).

Question
Find all points of discontinuity of \( f \), where \( f \) is defined by

\[
\begin{cases} 
|x|+3, & \text{if } x \leq -3 \\
-2x, & \text{if } -3 < x < 3 \\
6x+2, & \text{if } x \geq 3 
\end{cases}
\]

Answer:

The given function \( f \) is \( f(x) = \begin{cases} 
|x|+3 = -x+3, & \text{if } x \leq -3 \\
-2x, & \text{if } -3 < x < 3 \\
6x+2, & \text{if } x \geq 3 
\end{cases} \)

The given function \( f \) is defined at all the points of the real line.
Let \( c \) be a point on the real line.

Case I:
If \( c < -3 \), then \( f(c) = -c+3 \)

\[
\lim_{x \to c^-} f(x) = \lim_{x \to c^-} (-x+3) = -c+3
\]

\[
\lim_{x \to c^+} f(x) = \lim_{x \to c^+} (6x+2) = 6c+2
\]

\[
\therefore \lim_{x \to c} f(x) = f(c)
\]
Therefore, $f$ is continuous at all points $x$, such that $x < -3$

Case II:

If $c = -3$, then $f (-3) = -(-3) + 3 = 6$

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} (-x + 3) = -(-3) + 3 = 6$$

$$\lim_{x \to -3} f(x) = \lim_{x \to -3} (-2x) = -2x(-3) = 6$$

$$\therefore \lim_{x \to -3} f(x) = f(-3)$$

Therefore, $f$ is continuous at $x = -3$

Case III:

If $-3 < c < 3$, then $f(c) = -2c$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (-2x) = -2c$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, $f$ is continuous in $( -3, 3 )$.

Case IV:

If $c = 3$, then the left hand limit of $f$ at $x = 3$ is,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-2x) = -2 \times 3 = -6$$

The right hand limit of $f$ at $x = 3$ is,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x + 2) = 6 \times 3 + 2 = 20$$

It is observed that the left and right hand limit of $f$ at $x = 3$ do not coincide.

Therefore, $f$ is not continuous at $x = 3$

Case V:

If $c > 3$, then $f(c) = 6c + 2$ and $\lim_{x \to -c} f(x) = \lim_{x \to -c} (6x + 2) = 6c + 2$

$$\therefore \lim_{x \to -c} f(x) = f(c)$$

Therefore, $f$ is continuous at all points $x$, such that $x > 3$

Hence, $x = 3$ is the only point of discontinuity of $f$. 
Question

Find all points of discontinuity of $f$, where $f$ is defined by

$$f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Answer:

The given function $f$ is $f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

It is known that, $x < 0 \Rightarrow |x| = -x$ and $x > 0 \Rightarrow |x| = x$

Therefore, the given function can be rewritten as

$$f(x) = \begin{cases} \frac{|x|}{x} = \frac{-x}{x} = -1 & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ \frac{|x|}{x} = \frac{x}{x} = 1, & \text{if } x > 0 \end{cases}$$

The given function $f$ is defined at all the points of the real line.
Let \( c \) be a point on the real line.

Case I:

If \( c < 0 \), then \( f(c) = -1 \)
\[
\lim_{x \to c} f(x) = \lim_{x \to c} (-1) = -1
\]
\[
\therefore \lim_{x \to c} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points \( x < 0 \)

Case II:

If \( c = 0 \), then the left hand limit of \( f \) at \( x = 0 \) is,
\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1
\]
The right hand limit of \( f \) at \( x = 0 \) is,
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1) = 1
\]
It is observed that the left and right hand limit of \( f \) at \( x = 0 \) do not coincide.
Therefore, \( f \) is not continuous at \( x = 0 \)

Case III:

If \( c > 0 \), then \( f(c) = 1 \)
\[
\lim_{x \to c} f(x) = \lim_{x \to c} (1) = 1
\]
\[
\therefore \lim_{x \to c} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points \( x \), such that \( x > 0 \)

Hence, \( x = 0 \) is the only point of discontinuity of \( f \).

Question
Find all points of discontinuity of \( f \), where \( f \) is defined by

\[
f(x) = \begin{cases} 
\frac{x}{|x|}, & \text{if } x < 0 \\
-1, & \text{if } x \geq 0 
\end{cases}
\]

Answer:

The given function \( f \) is \( f(x) = \begin{cases} 
\frac{x}{|x|}, & \text{if } x < 0 \\
-1, & \text{if } x \geq 0 
\end{cases} \)

It is known that, \( x < 0 \Rightarrow |x| = -x \)

Therefore, the given function can be rewritten as

\[
f(x) = \begin{cases} 
\frac{x}{|x|} = \frac{x}{-x} = -1, & \text{if } x < 0 \\
-1, & \text{if } x \geq 0 
\end{cases}
\]

\( \Rightarrow f(x) = -1 \) for all \( x \in \mathbb{R} \)

Let \( c \) be any real number. Then, \( \lim_{x \to c} f(x) = \lim_{x \to c} (-1) = -1 \)

Also, \( f(c) = -1 = \lim_{x \to c} f(x) \)

Therefore, the given function is a continuous function.

Hence, the given function has no point of discontinuity.

Question
Find all points of discontinuity of $f$, where $f$ is defined by

$$f(x) = \begin{cases} 
  x+1, & \text{if } x \geq 1 \\
  x^2 + 1, & \text{if } x < 1 
\end{cases}$$

**Answer:**

The given function $f$ is defined as

$$f(x) = \begin{cases} 
  x+1, & \text{if } x \geq 1 \\
  x^2 + 1, & \text{if } x < 1 
\end{cases}$$

The given function $f$ is defined at all the points of the real line.

Let $c$ be a point on the real line.

**Case I:**

If $c < 1$, then $f(c) = c^2 + 1$ and

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^2 + 1) = c^2 + 1$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, $f$ is continuous at all points $x$, such that $x < 1$

**Case II:**

If $c = 1$, then $f(c) = f(1) = 1 + 1 = 2$
The left hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 1^2 + 1 = 2
\]
The right hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 1 + 1 = 2
\]
\[
\therefore \lim_{x \to 1} f(x) = f(1)
\]
Therefore, \( f \) is continuous at \( x = 1 \)

Case III:

If \( c > 1 \), then \( f(c) = c + 1 \)
\[
\lim_{x \to c} f(x) = \lim_{x \to c} (x + 1) = c + 1
\]
\[
\therefore \lim_{x \to c} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points \( x \), such that \( x > 1 \)

Hence, the given function \( f \) has no point of discontinuity.

Question
Find all points of discontinuity of $f$, where $f$ is defined by

$$f(x) = \begin{cases} 
  x^3 - 3, & \text{if } x \leq 2 \\
  x^2 + 1, & \text{if } x > 2
\end{cases}$$

Answer:

The given function $f$ is $f(x) = \begin{cases} 
  x^3 - 3, & \text{if } x \leq 2 \\
  x^2 + 1, & \text{if } x > 2
\end{cases}$

The given function $f$ is defined at all the points of the real line.

Let $c$ be a point on the real line.

Case I:
If $c < 2$, then $f(c) = c^3 - 3$ and $\lim_{{x \to c}} f(x) = \lim_{{x \to c}} (x^3 - 3) = c^3 - 3$

$\therefore \lim_{{x \to c}} f(x) = f(c)$

Therefore, $f$ is continuous at all points $x$, such that $x < 2$.
Case II:
If \( c = 2 \), then \( f(c) = f(2) = 2^3 - 3 = 5 \)
\[
\lim_{{x \to 2}} f(x) = \lim_{{x \to 2}} (x^3 - 3) = 2^3 - 3 = 5
\]
\[
\lim_{{x \to 2}} f(x) = \lim_{{x \to 2}} (x^2 + 1) = 2^2 + 1 = 5
\]
\[
\therefore \lim_{{x \to 2}} f(x) = f(2)
\]
Therefore, \( f \) is continuous at \( x = 2 \)

Case III:
If \( c > 2 \), then \( f(c) = c^3 + 1 \)
\[
\lim_{{x \to c}} f(x) = \lim_{{x \to c}} (x^2 + 1) = c^3 + 1
\]
\[
\therefore \lim_{{x \to c}} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points \( x \), such that \( x > 2 \)
Thus, the given function is continuous at every point on the real line.
Hence, \( f \) has no point of discontinuity.

Question
Find all points of discontinuity of \( f \), where \( f \) is defined by

\[
f(x) = \begin{cases} 
  x^{10} - 1, & \text{if } x \leq 1 \\
  x^2, & \text{if } x > 1
\end{cases}
\]

**Answer:**

The given function \( f(x) = \begin{cases} 
  x^{10} - 1, & \text{if } x \leq 1 \\
  x^2, & \text{if } x > 1
\end{cases} \)

The given function \( f \) is defined at all the points of the real line.

Let \( c \) be a point on the real line.

**Case I:**

If \( c < 1 \), then \( f(c) = c^{10} - 1 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (x^{10} - 1) = c^{10} - 1 \)

\[
\therefore \lim_{x \to c} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all points \( x \), such that \( x < 1 \)

**Case II:**

If \( c = 1 \), then the left hand limit of \( f \) at \( x = 1 \) is,

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^{10} - 1) = 1^{10} - 1 = 0
\]

The right hand limit of \( f \) at \( x = 1 \) is,

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2) = 1^2 = 1
\]

It is observed that the left and right hand limit of \( f \) at \( x = 1 \) do not coincide.

Therefore, \( f \) is not continuous at \( x = 1 \)

**Case III:**

If \( c > 1 \), then \( f(c) = c^2 \)

\[
\lim_{x \to c} f(x) = \lim_{x \to c} (x^2) = c^2
\]

\[
\therefore \lim_{x \to c} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 1 \)

Thus, from the above observation, it can be concluded that \( x = 1 \) is the only point of discontinuity of \( f \).
Question

Is the function defined by

\[ f(x) = \begin{cases} 
  x + 5, & \text{if } x \leq 1 \\
  x - 5, & \text{if } x > 1 
\end{cases} \]

a continuous function?

Answer:

The given function is

\[ f(x) = \begin{cases} 
  x + 5, & \text{if } x \leq 1 \\
  x - 5, & \text{if } x > 1 
\end{cases} \]

The given function \( f \) is defined at all the points of the real line.

Let \( c \) be a point on the real line.

Case I:

If \( c < 1 \), then \( f(c) = c + 5 \) and

\[ \lim_{{x \to c}} f(x) = \lim_{{x \to c}} (x + 5) = c + 5 \]

\[ \therefore \lim_{{x \to c}} f(x) = f(c) \]

Therefore, \( f \) is continuous at all points \( x \), such that \( x < 1 \).
Case II:
If \( c = 1 \), then \( f(1) = 1 + 5 = 6 \)

The left hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 5) = 1 + 5 = 6
\]

The right hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x - 5) = 1 - 5 = -4
\]

It is observed that the left and right hand limit of \( f \) at \( x = 1 \) do not coincide.

Therefore, \( f \) is not continuous at \( x = 1 \)

Case III:
If \( c > 1 \), then \( f(c) = c - 5 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (x - 5) = c - 5 \)

\[
\therefore \lim_{x \to c} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 1 \)

Thus, from the above observation, it can be concluded that \( x = 1 \) is the only point of discontinuity of \( f \).

Question
Discuss the continuity of the function $f$, where $f$ is defined by

$$f(x) = \begin{cases} 
3, & \text{if } 0 \leq x \leq 1 \\
4, & \text{if } 1 < x < 3 \\
5, & \text{if } 3 \leq x \leq 10 
\end{cases}$$

Answer:

The given function is $f(x) = \begin{cases} 
3, & \text{if } 0 \leq x \leq 1 \\
4, & \text{if } 1 < x < 3 \\
5, & \text{if } 3 \leq x \leq 10 
\end{cases}$

The given function is defined at all points of the interval $[0, 10]$. Let $c$ be a point in the interval $[0, 10]$.

Case 1: If $0 \leq c < 1$, then $f(c) = 3$ and $\lim_{x \to c} f(x) = \lim_{x \to c} 3 = 3$

\[\therefore \lim_{x \to c} f(x) = f(c)\]

Therefore, $f$ is continuous in the interval $[0, 1)$. 
Case II:
If \( c = 1 \), then \( f(c) = 3 \)
The left hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{{x \to 1^-}} f(x) = \lim_{{x \to 1^-}} (3) = 3
\]
The right hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{{x \to 1^+}} f(x) = \lim_{{x \to 1^+}} (4) = 4
\]
It is observed that the left and right hand limits of \( f \) at \( x = 1 \) do not coincide.
Therefore, \( f \) is not continuous at \( x = 1 \)

Case III:
If \( 1 < c < 3 \), then \( f(c) = 4 \) and 
\[
\lim_{{x \to c^-}} f(x) = \lim_{{x \to c^-}} (4) = 4
\]
\[
\lim_{{x \to c^+}} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points of the interval \((1, 3)\).
Case IV:
If $c = 3$, then $f(c) = 5$

The left hand limit of $f$ at $x = 3$ is,
$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (4) = 4$$

The right hand limit of $f$ at $x = 3$ is,
$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (5) = 5$$

It is observed that the left and right hand limits of $f$ at $x = 3$ do not coincide.
Therefore, $f$ is not continuous at $x = 3$

Case V:
If $3 < c \leq 10$, then $f(c) = 5$ and $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = 5$

$$\lim_{x \to c^-} f(x) = f(c)$$

Therefore, $f$ is continuous at all points of the interval $(3, 10]$.

Hence, $f$ is not continuous at $x = 1$ and $x = 3$
Discuss the continuity of the function $f$, where $f$ is defined by

$$f(x) = \begin{cases} 
2x, & \text{if } x < 0 \\
0, & \text{if } 0 \leq x \leq 1 \\
4x, & \text{if } x > 1
\end{cases}$$

Answer:

The given function is $f(x) = \begin{cases} 
2x, & \text{if } x < 0 \\
0, & \text{if } 0 \leq x \leq 1 \\
4x, & \text{if } x > 1
\end{cases}$

The given function is defined at all points of the real line.
Let $c$ be a point on the real line.

Case I:
If $c < 0$, then $f(c) = 2c$
$$\lim_{{x \to c}} f(x) = \lim_{{x \to c}} (2x) = 2c$$
$$\therefore \lim_{{x \to c}} f(x) = f(c)$$
Therefore, \( f \) is continuous at all points \( x \), such that \( x < 0 \)

**Case II:**

If \( c = 0 \), then \( f(c) = f(0) = 0 \)

The left hand limit of \( f \) at \( x = 0 \) is,

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x) = 2 \times 0 = 0
\]

The right hand limit of \( f \) at \( x = 0 \) is,

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (0) = 0
\]

\[
\therefore \lim_{x \to 0} f(x) = f(0)
\]

Therefore, \( f \) is continuous at \( x = 0 \)

**Case III:**

If \( 0 < c < 1 \), then \( f(c) = 0 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (0) = 0 \)

\[
\therefore \lim_{x \to c} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all points of the interval \((0, 1)\).
Case IV:
If \( c = 1 \), then \( f(c) = f(1) = 0 \)
The left hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 0 = 0
\]
The right hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x) = 4 \times 1 = 4
\]
It is observed that the left and right hand limits of \( f \) at \( x = 1 \) do not coincide.
Therefore, \( f \) is not continuous at \( x = 1 \)

Case V:
If \( c < 1 \), then \( f(c) = 4c \) and \( \lim_{x \to c^-} f(x) = \lim_{x \to c^-} (4x) = 4c \)
\[
\therefore \lim_{x \to c^-} f(x) = f(c)
\]
Therefore, \( f \) is continuous at all points \( x \), such that \( x > 1 \)
Hence, \( f \) is not continuous only at \( x = 1 \)

Question
Discuss the continuity of the function \( f \), where \( f \) is defined by

\[
f(x) = \begin{cases} 
-2, & \text{if } x \leq -1 \\
2x, & \text{if } -1 < x \leq 1 \\
2, & \text{if } x > 1 
\end{cases}
\]

Answer:

The given function \( f \) is defined as

\[
f(x) = \begin{cases} 
-2, & \text{if } x \leq -1 \\
2x, & \text{if } -1 < x \leq 1 \\
2, & \text{if } x > 1 
\end{cases}
\]

The given function is defined at all points of the real line.

Let \( c \) be a point on the real line.

Case I:

If \( c < -1 \), then \( f(c) = -2 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (-2) = -2 \)

\( \therefore \lim_{x \to c} f(x) = f(c) \)

Therefore, \( f \) is continuous at all points \( x \), such that \( x < -1 \)
Case II:
If \( c = -1 \), then \( f(c) = f(-1) = -2 \)

The left hand limit of \( f \) at \( x = -1 \) is,
\[
\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-2) = -2
\]
The right hand limit of \( f \) at \( x = -1 \) is,
\[
\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (2x) = 2(-1) = -2
\]
\[\therefore \lim_{x \to -1} f(x) = f(-1)\]

Therefore, \( f \) is continuous at \( x = -1 \)

Case III:
If \(-1 < c < 1\), then \( f(c) = 2c \)

\[
\lim_{x \to c} f(x) = \lim_{x \to c} (2x) = 2c
\]
\[\therefore \lim_{x \to c} f(x) = f(c)\]

Therefore, \( f \) is continuous at all points of the interval \((-1, 1)\).
Case IV:
If \( c = 1 \), then \( f(1) = 2 \times 1 = 2 \)

The left hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x) = 2 \times 1 = 2
\]

The right hand limit of \( f \) at \( x = 1 \) is,
\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 = 2
\]
\[
\therefore \lim_{x \to 1} f(x) = f(1)
\]

Therefore, \( f \) is continuous at \( x = 1 \)

Case V:
If \( c > 1 \), then \( f(c) = 2 \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} (2x) = 2 \)

\[
\lim_{x \to c} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all points \( x \), such that \( x > 1 \)

Thus, from the above observations, it can be concluded that \( f \) is continuous at all points of the real line.

Question
Find the relationship between \( a \) and \( b \) so that the function \( f \) defined by

\[
f(x) = \begin{cases} 
ax + 1, & \text{if } x \leq 3 \\
bx + 3, & \text{if } x > 3 
\end{cases}
\]

is continuous at \( x = 3 \).

Answer:

The given function is \( f(x) = \begin{cases} 
ax + 1, & \text{if } x \leq 3 \\
bx + 3, & \text{if } x > 3 
\end{cases} \)

If \( f \) is continuous at \( x = 3 \), then

\[
\lim_{{x \to 3^-}} f(x) = \lim_{{x \to 3^+}} f(x) = f(3) \quad \ldots(1)
\]

Also,

\[
\lim_{{x \to 3^-}} f(x) = \lim_{{x \to 3^-}} (ax + 1) = 3a + 1
\]

\[
\lim_{{x \to 3^+}} f(x) = \lim_{{x \to 3^+}} (bx + 3) = 3b + 3
\]

\[
f(3) = 3a + 1
\]

Therefore, from (1), we obtain

\[
3a + 1 = 3b + 3 = 3a + 1
\]

\[
\Rightarrow 3a + 1 = 3b + 3
\]

\[
\Rightarrow 3a = 3b + 2
\]

\[
\Rightarrow a = b + \frac{2}{3}
\]

Therefore, the required relationship is given by, \( a = b + \frac{2}{3} \)
For what value of \( \lambda \) is the function defined by

\[
    f(x) = \begin{cases} 
        \lambda(x^2 - 2x), & \text{if } x \leq 0 \\ 
        4x + 1, & \text{if } x > 0 
    \end{cases}
\]

continuous at \( x = 0 \)? What about continuity at \( x = 1 \)?

Answer:

The given function is \( f(x) = \begin{cases} 
    \lambda(x^2 - 2x), & \text{if } x \leq 0 \\ 
    4x + 1, & \text{if } x > 0 
    \end{cases} \)

If \( f \) is continuous at \( x = 0 \), then

\[
    \lim_{x \to 0} f(x) = \lim_{x \to 0} \lambda(x^2 - 2x) = \lambda(0^2 - 2 \times 0)
\]

\[
    \lim_{x \to 0} f(x) = \lim_{x \to 0} (4x + 1) = 4 \times 0 + 1 = 0
\]

\[
    \Rightarrow \lambda(0) = 0 = 0, \text{ which is not possible}
\]

Therefore, there is no value of \( \lambda \) for which \( f \) is continuous at \( x = 0 \)

At \( x = 1 \),

\[
    f(1) = 4 \times 1 + 1 = 4 \times 1 + 1 = 5
\]

\[
    \lim_{x \to 1} (4x + 1) = 4 \times 1 + 1 = 5
\]

\[
    \therefore \lim_{x \to 1} f(x) = f(1)
\]

Therefore for all values of \( \lambda \) \( f \) is continuous at \( x = 1 \)

Question
Hence \( g \) is discontinuous at all integral points

**Question**
Is the function defined by \( f(x) = x^2 - \sin x + 5 \) continuous at \( x = \pi \)?

**Answer:**

The given function is \( f(x) = x^2 - \sin x + 5 \).

It is evident that \( f \) is defined at \( x = \pi \).

At \( x = \pi \), \( f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5 \)

Consider \( \lim_{x \to \pi} f(x) = \lim_{x \to \pi} (x^2 - \sin x + 5) \)

Put \( x = \pi + h \)

If \( x \to \pi \), then it is evident that \( h \to 0 \)

\[ \therefore \lim_{x \to \pi} f(x) = \lim_{x \to \pi} (x^2 - \sin x + 5) \]

\[ = \lim_{h \to 0} \left[ (\pi + h)^2 - \sin (\pi + h) + 5 \right] \]

\[ = \lim_{h \to 0} (\pi + h)^2 - \lim_{h \to 0} \sin (\pi + h) + \lim_{h \to 0} 5 \]

\[ = (\pi + 0)^2 - \lim_{h \to 0} [\sin \pi \cosh + \cos \pi \sinh] + 5 \]

\[ = \pi^2 - \lim_{h \to 0} \sin \pi \cosh - \lim_{h \to 0} \cos \pi \sinh + 5 \]

\[ = \pi^2 - \sin \pi \cos 0 - \cos \pi \sin 0 + 5 \]

\[ = \pi^2 - 0 \times 1 - (-1) \times 0 + 5 \]

\[ = \pi^2 + 5 \]

\[ \therefore \lim_{x \to \pi} f(x) = f(\pi) \]

Therefore, the given function \( f \) is continuous at \( x = \pi \)

**Question**
Discuss the continuity of the following functions.

(a) \( f(x) = \sin x + \cos x \)
(b) \( f(x) = \sin x - \cos x \)
(c) \( f(x) = \sin x \times \cos x \)

Answer:

It is known that if \( g \) and \( h \) are two continuous functions, then \( g + h, g - h, \) and \( g \cdot h \) are also continuous.

It has to proved first that \( g(x) = \sin x \) and \( h(x) = \cos x \) are continuous functions.

Let \( g(x) = \sin x \)

It is evident that \( g(x) = \sin x \) is defined for every real number.

Let \( c \) be a real number. Put \( x = c + h \)

If \( x \to c \), then \( h \to 0 \)

\[
\lim_{x \to c} g(x) = \lim_{x \to c} \sin x
\]
\[
\begin{align*}
= \lim_{h \to 0} \sin(c + h) \\
= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h] \\
= \lim_{h \to 0} (\sin c \cos h) + \lim_{h \to 0} (\cos c \sin h) \\
= \sin c \cos 0 + \cos c \sin 0 \\
= \sin c + 0 \\
= \sin c \\
\end{align*}
\]
\[\therefore \lim_{x \to c} g(x) = g(c)\]

Therefore, \(g\) is a continuous function.

Let \(h(x) = \cos x\)

It is evident that \(h(x) = \cos x\) is defined for every real number.

Let \(c\) be a real number. Put \(x = c + h\)

If \(x \to c \neq c\), then \(h \to c \neq 0\)

\(h(c) = \cos c\)

\[\lim_{x \to c} h(x) = \lim_{x \to c} \cos x\]
\[= \lim_{h \to 0} \cos(c + h)\]
\[= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]\]
\[= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h\]
\[= \cos c \cos 0 - \sin c \sin 0\]
\[= \cos c \times 1 - \sin c \times 0\]
\[= \cos c\]

\[\therefore \lim_{x \to c} h(x) = h(c)\]

Therefore, \(h\) is a continuous function.

Therefore, it can be concluded that

(a) \(f(x) = g(x) + h(x) = \sin x + \cos x\) is a continuous function

(b) \(f(x) = g(x) - h(x) = \sin x - \cos x\) is a continuous function

(c) \(f(x) = g(x) \times h(x) = \sin x \times \cos x\) is a continuous function
Question

Discuss the continuity of the cosine, cosecant, secant and cotangent functions,

Answer:

It is known that if \( g \) and \( h \) are two continuous functions, then

\[
(i) \quad \frac{h(x)}{g(x)}, \quad g(x) \neq 0 \quad \text{is continuous}
\]

\[
(ii) \quad \frac{1}{g(x)}, \quad g(x) \neq 0 \quad \text{is continuous}
\]

\[
(iii) \quad \frac{1}{h(x)}, \quad h(x) \neq 0 \quad \text{is continuous}
\]

It has to be proved first that \( g(x) = \sin x \) and \( h(x) = \cos x \) are continuous functions.

Let \( g(x) = \sin x \)

It is evident that \( g(x) = \sin x \) is defined for every real number.

Let \( c \) be a real number. Put \( x = c + h \)

If \( x \to c \), then \( h \to 0 \)

\[
g(c) = \sin c
\]
\[
\lim_{x \to c} g(x) = \lim_{h \to 0} \sin(x + h) = \lim_{h \to 0} [\sin(c \cos h + \cos c \sin h)] \\
= \lim_{h \to 0} (\sin(c \cos h) + \cos c \sin h) \\
= \sin c \cos 0 + \cos c \sin 0 \\
= \sin c + 0 \\
= \sin c
\]

Therefore, \( g \) is a continuous function.

Let \( h(x) = \cos x \)

It is evident that \( h(x) = \cos x \) is defined for every real number.

Let \( c \) be a real number. Put \( x = c + h \)

If \( x \to c \), then \( h \to 0 \)

\( h(c) = \cos c \)
\[
\lim_{x \to c} h(x) = \lim_{x \to c} \cos x
\]
\[
= \lim_{h \to 0} \cos (c + h)
\]
\[
= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]
\]
\[
= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h
\]
\[
= \cos c \cdot 1 - \sin c \cdot 0
\]
\[
= \cos c
\]
\[
\therefore \lim_{x \to c} h(x) = h(c)
\]

Therefore, \( h(x) = \cos x \) is continuous function.

It can be concluded that,

\[
\csc x = \frac{1}{\sin x}, \quad \sin x \neq 0 \text{ is continuous}
\]

\[
\Rightarrow \csc x, \quad x \neq n\pi \quad (n \in \mathbb{Z}) \text{ is continuous}
\]

Therefore, cosecant is continuous except at \( x = n\pi, \quad n \in \mathbb{Z} \)

\[
\sec x = \frac{1}{\cos x}, \quad \cos x \neq 0 \text{ is continuous}
\]

\[
\Rightarrow \sec x, \quad x \neq \left(2n + 1\right)\frac{\pi}{2} \quad (n \in \mathbb{Z}) \text{ is continuous}
\]

Therefore, secant is continuous except at \( x = \left(2n + 1\right)\frac{\pi}{2}, \quad n \in \mathbb{Z} \)

\[
\cot x = \frac{\cos x}{\sin x}, \quad \sin x \neq 0 \text{ is continuous}
\]

\[
\Rightarrow \cot x, \quad x \neq n\pi \quad (n \in \mathbb{Z}) \text{ is continuous}
\]

Therefore, cotangent is continuous except at \( x = n\pi, \quad n \in \mathbb{Z} \)

Question
Find the points of discontinuity of \( f \), where

\[
f(x) = \begin{cases} 
\sin(x)/x, & \text{if } x < 0 \\
x + 1, & \text{if } x \geq 0 
\end{cases}
\]

Answer:

The given function \( f(x) = \begin{cases} \sin(x)/x, & \text{if } x < 0 \\
x + 1, & \text{if } x \geq 0 
\end{cases} \)

It is evident that \( f \) is defined at all points of the real line.

Let \( c \) be a real number.

Case I:

If \( c < 0 \), then

\[
\lim_{x \to c} f(x) = \lim_{x \to c} \left( \frac{\sin(x)}{x} \right) = \frac{\sin(c)}{c}
\]

\[
\therefore \lim_{x \to c} f(x) = f(c)
\]

Therefore, \( f \) is continuous at all points \( x \), such that \( x < 0 \).
Thus \( f \) has no point of discontinuity

**Question**
Determine if \( f \) defined by
\[
 f(x) = \begin{cases} 
 x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\
 0, & \text{if } x = 0
\end{cases}
\]
is a continuous function?

Answer:

The given function \( f \) is
\[
 f(x) = \begin{cases} 
 x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\
 0, & \text{if } x = 0
\end{cases}
\]

It is evident that \( f \) is defined at all points of the real line.

Let \( c \) be a real number.

Case I:

If \( c \neq 0 \), then \( f(c) = c^2 \sin \frac{1}{c} \)

\[
\lim_{x \to c} f(x) = \lim_{x \to c} \left( x^2 \sin \frac{1}{x} \right) = \left( \lim_{x \to c} x^2 \right) \left( \lim_{x \to c} \sin \frac{1}{x} \right) = c^2 \sin \frac{1}{c}
\]
Therefore, \( f \) is continuous at all points \( x \neq 0 \)

Case II:

If \( c = 0 \), then \( f(0) = 0 \)

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} \left( x^2 \sin \frac{1}{x} \right) = \lim_{x \to 0} \left( x^2 \sin \frac{1}{x} \right)
\]

It is known that, \( -1 \leq \sin \frac{1}{x} \leq 1, \ x \neq 0 \)

\[
\Rightarrow -x^2 \leq \sin \frac{1}{x} \leq x^2
\]

\[
\Rightarrow \lim_{x \to 0} (-x^2) \leq \lim_{x \to 0} \left( x^2 \sin \frac{1}{x} \right) \leq \lim_{x \to 0} x^2
\]

\[
\Rightarrow 0 \leq \lim_{x \to 0} \left( x^2 \sin \frac{1}{x} \right) \leq 0
\]

\[
\Rightarrow \lim_{x \to 0} \left( x^2 \sin \frac{1}{x} \right) = 0
\]

\[
\therefore \lim_{x \to 0} f(x) = 0
\]

Similarly, \( \lim_{x \to 0} f(x) = \lim_{x \to 0} \left( x^2 \sin \frac{1}{x} \right) = \lim_{x \to 0} \left( x^2 \sin \frac{1}{x} \right) = 0 \)

\[
\therefore \lim_{x \to 0} f(x) = f(0) = \lim_{x \to 0} f(x)
\]

Therefore, \( f \) is continuous at \( x = 0 \)

From the above observations, it can be concluded that \( f \) is continuous at every point of the real line.

Thus, \( f \) is a continuous function.

Question
Examine the continuity of $f$, where $f$ is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

Answer:

The given function $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$

It is evident that $f$ is defined at all points of the real line.

Let $c$ be a real number.

Case I:

If $c \neq 0$, then $f(c) = \sin c - \cos c$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (\sin x - \cos x) = \sin c - \cos c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, $f$ is continuous at all points $x$, such that $x \neq 0$

Case II:

If $c = 0$, then $f(0) = -1$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (\sin x - \cos x) = 0 - \cos 0 = 0 - 1 = -1$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (\sin x - \cos x) = 0 - \cos 0 = 0 - 1 = -1$$

$$\therefore \lim_{x \to 0} f(x) = \lim_{x \to 0} f(x) = f(0)$$

Therefore, $f$ is continuous at $x = 0$

From the above observations, it can be concluded that $f$ is continuous at every point of the real line.

Thus, $f$ is a continuous function.

Question
Find the values of $k$ so that the function $f$ is continuous at the indicated point.

$$f(x) = \begin{cases} 
\frac{k\cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\
3, & \text{if } x = \frac{\pi}{2}
\end{cases} \quad \text{at } x = \frac{\pi}{2}$$

Answer:

The given function $f(x) = \begin{cases} 
\frac{k\cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\
3, & \text{if } x = \frac{\pi}{2}
\end{cases}$

The given function $f$ is continuous at $x = \frac{\pi}{2}$ if $f$ is defined at $x = \frac{\pi}{2}$ and if the value of the limit at $x = \frac{\pi}{2}$ equals the limit of $f$ at $x = \frac{\pi}{2}$.

It is evident that $f$ is defined at $x = \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = 3$.

$$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k\cos x}{\pi - 2x}$$

Put $x = \frac{\pi}{2} + h$

Then, $x \to \frac{\pi}{2} \Rightarrow h \to 0$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k\cos x}{\pi - 2x} = \lim_{h \to 0} \frac{k\cos \left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = k \lim_{h \to 0} \frac{-\sin h}{-2h} = k \lim_{h \to 0} \frac{\sin h}{2h} = k \cdot \frac{1}{2}$$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$\Rightarrow \frac{k}{2} = 3$

$\Rightarrow k = 6$

Therefore, the required value of $k$ is 6.
Question

Find the values of $k$ so that the function $f$ is continuous at the indicated point.

$$f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases} \quad \text{at } x = 2$$

Answer:

The given function is $f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases}$

The given function $f$ is continuous at $x = 2$, if $f$ is defined at $x = 2$ and if the value of $f$ at $x = 2$ equals the limit of $f$ at $x = 2$.

It is evident that $f$ is defined at $x = 2$ and $f(2) = k(2)^2 = 4k$.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} kx^2 = k(2)^2 = 4k$$

\[\Rightarrow k \times 2^2 = 4k\]
\[\Rightarrow 4k = 3 \Rightarrow 4k = 3\]
\[\Rightarrow k = \frac{3}{4}\]

Therefore, the required value of $k$ is $\frac{3}{4}$.

Question
Find the values of \( k \) so that the function \( f \) is continuous at the indicated point.

\[
f(x) = \begin{cases} 
kx + 1, & \text{if } x \leq \pi \\
\cos x, & \text{if } x > \pi
\end{cases}
\] at \( x = \pi \)

Answer:

The given function is

\[
f(x) = \begin{cases} 
kx + 1, & \text{if } x \leq \pi \\
\cos x, & \text{if } x > \pi
\end{cases}
\]

The given function \( f \) is continuous at \( x = \pi \), if \( f \) is defined at \( x = \pi \) and if the value of \( \lim_{x \to \pi} f(x) \) equals the limit of \( f(x) \) at \( x = \pi \).

It is evident that \( f \) is defined at \( x = \pi \) and \( f(\pi) = k\pi + 1 \).

\[
\lim_{x \to \pi} f(x) = \lim_{x \to \pi} f(x) = f(\pi)
\]

\[
\Rightarrow \lim_{x \to \pi} (kx + 1) = \lim_{x \to \pi} \cos x = k\pi + 1
\]

\[
\Rightarrow k\pi + 1 = \cos \pi = k\pi + 1
\]

\[
\Rightarrow k\pi + 1 = -1 = k\pi + 1
\]

\[
\Rightarrow k = -\frac{2}{\pi}
\]

Therefore, the required value of \( k \) is \( -\frac{2}{\pi} \).

Question

Find the values of \( k \) so that the function \( f \) is continuous at the indicated point.

\[
f(x) = \begin{cases} 
kx + 1, & \text{if } x \leq 5 \\
3x - 5, & \text{if } x > 5
\end{cases}
\] at \( x = 5 \)

Answer:

The given function \( f \) is

\[
f(x) = \begin{cases} 
kx + 1, & \text{if } x \leq 5 \\
3x - 5, & \text{if } x > 5
\end{cases}
\]

The given function \( f \) is continuous at \( x = 5 \), if \( f \) is defined at \( x = 5 \) and if the value of \( \lim_{x \to 5} f(x) \) equals the limit of \( f(x) \) at \( x = 5 \).

It is evident that \( f \) is defined at \( x = 5 \) and \( f(5) = kx + 1 = 5k + 1 \).

\[
\lim_{x \to 5} f(x) = \lim_{x \to 5} f(x) = f(5)
\]

\[
\Rightarrow \lim_{x \to 5} (kx + 1) = \lim_{x \to 5} (3x - 5) = 5k + 1
\]

\[
\Rightarrow 5k + 1 = 15 - 5 = 5k + 1
\]

\[
\Rightarrow 5k + 1 = 10
\]

\[
\Rightarrow 5k = 9
\]
Therefore, the required value of \( k \) is \( \frac{9}{5} \).

Question

Find the values of \( a \) and \( b \) such that the function defined by

\[
f(x) = \begin{cases} 
5, & \text{if } x \leq 2 \\
ax + b, & \text{if } 2 < x < 10 \\
21, & \text{if } x \geq 10 
\end{cases}
\]

is a continuous function.

Answer:

The given function \( f(x) \) is defined as

\[
f(x) = \begin{cases} 
5, & \text{if } x \leq 2 \\
ax + b, & \text{if } 2 < x < 10 \\
21, & \text{if } x \geq 10 
\end{cases}
\]

It is evident that the given function \( f \) is defined at all points of the real line.
If \( f \) is a continuous function, then \( f \) is continuous at all real numbers.
In particular, \( f \) is continuous at \( x = 2 \) and \( x = 10 \)
Since \( f \) is continuous at \( x = 2 \), we obtain
\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} f(x) = f(2)
\]
\[
\Rightarrow \lim_{x \to 2} (5) = \lim_{x \to 2} (ax + b) = 5
\]
\[
\Rightarrow 5 = 2a + b = 5
\]
\[
\Rightarrow 2a + b = 5 \quad \ldots(1)
\]

Since \( f \) is continuous at \( x = 10 \), we obtain

\[
\lim_{x \to 10} f(x) = \lim_{x \to 10} f(x) = f(10)
\]
\[
\Rightarrow \lim_{x \to 10} (ax + b) = \lim_{x \to 10} (21) = 21
\]
\[
\Rightarrow 10a + b = 21 = 21
\]
\[
\Rightarrow 10a + b = 21 \quad \ldots(2)
\]

On subtracting equation (1) from equation (2), we obtain

\[
8a = 16
\]
\[
\Rightarrow a = 2
\]

By putting \( a = 2 \) in equation (1), we obtain

\[
2 \times 2 + b = 5
\]
\[
\Rightarrow 4 + b = 5
\]
\[
\Rightarrow b = 1
\]

Therefore, the values of \( a \) and \( b \) for which \( f \) is a continuous function are 2 and 1 respectively.

Question
Show that the function defined by \( f(x) = \cos(x^2) \) is a continuous function.

Answer:

The given function is \( f(x) = \cos(x^2) \).

This function is defined for every real number and can be written as the composition of two functions as,

\[ f = g \circ h, \text{ where } g(x) = \cos x \text{ and } h(x) = x^2 \]

\[ (\circ) \quad (g \circ h)(x) = g(h(x)) = g(x^2) = \cos(x^2) = f(x) \]

It has to be first proved that \( g(x) = \cos x \) and \( h(x) = x^2 \) are continuous functions.

It is evident that \( g(x) = \cos x \) is defined for every real number.

Let \( c \) be a real number.

Then, \( g(c) = \cos c \)

Put \( x = c + h \)

If \( x \to c \), then \( h \to 0 \)

\[ \lim_{x \to c} g(x) = \lim_{x \to c} \cos x \]

\[ = \lim_{h \to 0} \cos(c + h) \]

\[ = \lim_{h \to 0} \left[ \cos c \cos h - \sin c \sin h \right] \]

\[ = \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h \]

\[ = \cos c \cdot 1 - \sin c \cdot 0 \]

\[ = \cos c \]

\[ \therefore \lim_{x \to c} g(x) = g(c) \]

Therefore, \( g(x) = \cos x \) is a continuous function.

\( h(x) = x^2 \)

Clearly, \( h \) is defined for every real number.

Let \( k \) be a real number, then \( h(k) = k^2 \)

\[ \lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2 \]

\[ \therefore \lim_{x \to k} h(x) = h(k) \]

Therefore, \( h \) is a continuous function.

It is known that for real valued functions \( g \) and \( h \), such that \( (g \circ h) \) is defined at \( c \), if \( g \) is continuous at \( c \) and if \( f \) is continuous at \( g(c) \), then \( (f \circ g) \) is continuous at \( c \).
Therefore, \( f(x) = (g \circ h)(x) = \cos(x^2) \) is a continuous function.

**Question**

Show that the function defined by \( f(x) = |\cos x| \) is a continuous function.

**Answer:**

The given function is \( f(x) = |\cos x| \)

This function is defined for every real number and can be written as the composition of two functions as,

\[
 f = g \circ h, \text{ where } g(x) = |x| \text{ and } h(x) = \cos x
\]

\[
 \therefore (g \circ h)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)
\]

It has to be first proved that \( g(x) = |x| \) and \( h(x) = \cos x \) are continuous functions.

\( g(x) = |x| \) can be written as

\[
 g(x) = \begin{cases} 
 -x, & \text{if } x < 0 \\
 x, & \text{if } x \geq 0 
\end{cases}
\]

Clearly, \( g \) is defined for all real numbers.

Let \( c \) be a real number.
Case I:
If \( c < 0 \), then \( g(c) = -c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c \)
\[ \therefore \lim_{x \to c} g(x) = g(c) \]
Therefore, \( g \) is continuous at all points \( x \), such that \( x < 0 \)

Case II:
If \( c > 0 \), then \( g(c) = c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} x = c \)
\[ \therefore \lim_{x \to c} g(x) = g(c) \]
Therefore, \( g \) is continuous at all points \( x \), such that \( x > 0 \)

Case III:
If \( c = 0 \), then \( g(c) = g(0) = 0 \)
\[ \lim_{x \to 0} g(x) = \lim_{x \to 0} (-x) = 0 \]
\[ \lim_{x \to 0} g(x) = \lim_{x \to 0} (x) = 0 \]
\[ \therefore \lim_{x \to 0} g(x) = \lim_{x \to 0} (x) = g(0) \]
Therefore, \( g \) is continuous at \( x = 0 \)
From the above three observations, it can be concluded that $g(x)$ is continuous at all points. 

$h(x) = \cos x$

It is evident that $h(x) = \cos x$ is defined for every real number.

Let $c$ be a real number. Put $x = c + h$

If $x \to c$, then $h \to 0$

$h(c) = \cos c$

$$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$$
$$= \lim_{h \to 0} \cos (c + h)$$
$$= \lim_{h \to 0} [\cos c \cos h - \sin c \sin h]$$
$$= \lim_{h \to 0} \cos c \cos h - \lim_{h \to 0} \sin c \sin h$$
$$= \cos c \cos 0 - \sin c \sin 0$$
$$= \cos c \times 1 - \sin c \times 0$$
$$= \cos c$$

$\therefore \lim_{x \to c} h(x) = h(c)$

Therefore, $h(x) = \cos x$ is a continuous function.

It is known that for real valued functions $g$ and $h$, such that $(g \circ h)$ is defined at $c$, if $g$ is continuous at $c$ and if $f$ is continuous at $g(c)$, then $(f \circ g)$ is continuous at $c$.

Therefore, $f(x) = (g \circ h)(x) = g(h(x)) = g(\cos x) = |\cos x|$ is a continuous function.

Question
Examine that $\sin|x|$ is a continuous function.

Answer:

Let $f(x) = \sin|x|$

This function is defined for every real number and $f$ can be written as the composition of two functions as,

$f = g \circ h$, where $g(x) = |x|$ and $h(x) = \sin x$

$\therefore (g \circ h)(x) = g(h(x)) = g(\sin x) = |\sin x| = f(x)$

It has to be proved first that $g(x) = |x|$ and $h(x) = \sin x$ are continuous functions.

$g(x) = |x|$ can be written as

$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$

Clearly, $g$ is defined for all real numbers.

Let $c$ be a real number.
Case I:

If \( c < 0 \), then \( g(c) = -c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c \)

\[
\therefore \lim_{x \to c} g(x) = g(c)
\]

Therefore, \( g \) is continuous at all points \( x \), such that \( x < 0 \)

Case II:

If \( c > 0 \), then \( g(c) = c \) and \( \lim_{x \to c} g(x) = \lim_{x \to c} x = c \)

\[
\therefore \lim_{x \to c} g(x) = g(c)
\]

Therefore, \( g \) is continuous at all points \( x \), such that \( x > 0 \)

Case III:

If \( c = 0 \), then \( g(c) = g(0) = 0 \)

\[
\lim_{x \to 0} g(x) = \lim_{x \to 0} (-x) = 0 \\
\lim_{x \to 0} g(x) = \lim_{x \to 0} x = 0 \\
\therefore \lim_{x \to 0} g(x) = \lim_{x \to 0} x = g(0)
\]
Therefore, $g$ is continuous at $x = 0$

From the above three observations, it can be concluded that $g$ is continuous at all points.

$h(x) = \sin x$

It is evident that $h(x) = \sin x$ is defined for every real number.

Let $c$ be a real number. Put $x = c + k$

If $x \to c$, then $k \to 0$

$h(c) = \sin c$

\[
\lim_{x \to c} h(x) = \lim_{x \to c} \sin x = \lim_{x \to c} \sin(c + k) = \lim_{k \to 0} \left[ \sin c \cos k + \cos c \sin k \right] = \lim_{k \to 0} (\sin c \cos k) + \lim_{k \to 0} (\cos c \sin k) = \sin c \cos 0 + \cos c \sin 0 = \sin c + 0 = \sin c
\]

\[\therefore \lim_{x \to c} h(x) = g(c)\]

Therefore, $h$ is a continuous function.

It is known that for real valued functions $g$ and $h$, such that $(g \circ h)$ is defined at $c$, if $g$ is continuous at $c$ and if $f$ is continuous at $g(c)$, then $(f \circ g)$ is continuous at $c$.

Therefore, $f(x) = (g \circ h)(x) = g(h(x)) = g(\sin x) = |\sin x|$ is a continuous function.

Question
Find all the points of discontinuity of $f$ defined by $f(x) = |x| - |x + 1|$

Answer:
The given function is $f(x) = |x| - |x + 1|$
The two functions, $g$ and $h$, are defined as $g(x) = |x|$ and $h(x) = |x + 1|$
Then, $f = g - h$
The continuity of $g$ and $h$ is examined first.
$g(x) = |x|$ can be written as
$g(x) = \begin{cases} 
-x, & \text{if } x < 0 \\
0, & \text{if } x = 0 \\
x, & \text{if } x > 0
\end{cases}$
Clearly, $g$ is defined for all real numbers.
Let $c$ be a real number.
Case I:
If $c < 0$, then $g(c) = -c$ and $\lim_{x \to c} g(x) = \lim_{x \to -c} (-x) = -c$
\[\therefore \lim_{x \to c} g(x) = g(c)\]
Therefore, $g$ is continuous at all points $x$, such that $x < 0$

Case II:
If $c > 0$, then $g(c) = c$ and $\lim_{x \to c} g(x) = \lim_{x \to c} x = c$
\[\therefore \lim_{x \to c} g(x) = g(c)\]
Therefore, $g$ is continuous at all points $x$, such that $x > 0$

Case III:
If $c = 0$, then $g(c) = g(0) = 0$
\[\lim_{x \to 0} g(x) = \lim_{x \to 0} (-x) = 0\]
\[\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} x = 0\]
\[\therefore \lim_{x \to 0} g(x) = \lim_{x \to 0} g(x) = g(0)\]
Therefore, $g$ is continuous at $x = 0$
From the above three observations, it can be concluded that $g(x)$ is continuous at all points.

$h(x) = |x + 1|$ can be written as

$$h(x) = \begin{cases} 
-(x+1), & \text{if } x < -1 \\
(x+1), & \text{if } x \geq -1
\end{cases}$$

Clearly, $h(x)$ is defined for every real number.

Let $c$ be a real number.

Case I:

If $c < -1$, then $h(c) = -(c+1)$ and

$$\lim_{x \to c} h(x) = \lim_{x \to c} [- (x+1)] = -(c+1)$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, $h(x)$ is continuous at all points $x$, such that $x < -1$

Case II:

If $c > -1$, then $h(c) = c + 1$ and

$$\lim_{x \to c} h(x) = \lim_{x \to c} (x + 1) = c + 1$$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, $h(x)$ is continuous at all points $x$, such that $x > -1$

Case III:

If $c = -1$, then $h(c) = h(-1) = -1 + 1 = 0$

$$\lim_{x \to -1} h(x) = \lim_{x \to -1} [- (x+1)] = -(1+1) = 0$$

$$\lim_{x \to -1} h(x) = \lim_{x \to -1} (x+1) = (-1+1) = 0$$

$$\therefore \lim_{x \to -1} h(x) = \lim_{x \to -1} h(x) = h(-1)$$

Therefore, $h(x)$ is continuous at $x = -1$

From the above three observations, it can be concluded that $h(x)$ is continuous at all points of the real line.

Question
Verify Rolle's Theorem for the function \( f(x) = x^3 + 2x - 8, \ x \in [-4, 2] \)

Answer:

The given function, \( f(x) = x^3 + 2x - 8 \), being a polynomial function, is continuous in \([-4, 2]\) and is differentiable in \((-4, 2)\).

\[
\begin{align*}
  f(-4) &= (-4)^3 + 2(-4) - 8 = 16 - 8 - 8 = 0 \\
  f(2) &= (2)^3 + 2(2) - 8 = 8 + 4 - 8 = 0 \\
  \therefore \ f(-4) &= f(2) = 0
\end{align*}
\]

\( \Rightarrow \) The value of \( f(x) \) at -4 and 2 coincides.

Rolle's Theorem states that there is a point \( c \in (-4, 2) \) such that \( f'(c) = 0 \)

\[
\begin{align*}
  f(x) &= x^3 + 2x - 8 \\
  \Rightarrow \ f'(x) &= 3x^2 + 2 \\
  \therefore \ f'(c) &= 0 \\
  \Rightarrow \ 2c + 2 &= 0 \\
  \Rightarrow \ 2c + 2 &= 0 \\
  \Rightarrow \ c &= -1, \ \text{where} \ c = -1 \in (-4, 2)
\end{align*}
\]

Hence, Rolle's Theorem is verified for the given function.

Question
Examine if Rolle's Theorem is applicable to any of the following functions. Can you say something about the converse of Rolle's Theorem from these examples?

(i) \( f(x) = [x] \) for \( x \in [5, 9] \)

(ii) \( f(x) = [x] \) for \( x \in [-2, 2] \)

(iii) \( f(x) = x^3 - 1 \) for \( x \in [1, 2] \)

Answer:

By Rolle's Theorem, for a function \( f : [a, b] \to \mathbb{R} \), if

(a) \( f \) is continuous on \([a, b]\)

(b) \( f \) is differentiable on \((a, b)\)

(c) \( f(a) = f(b) \)

then, there exists some \( c \in (a, b) \) such that \( f'(c) = 0 \)

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) \( f(x) = [x] \) for \( x \in [5, 9] \)

It is evident that the given function \( f(x) \) is not continuous at every integral point.

In particular, \( f(x) \) is not continuous at \( x = 5 \) and \( x = 9 \)

\[ \Rightarrow f(x) \text{ is not continuous in } [5, 9]. \]

Also, \( f(5) = [5] = 5 \) and \( f(9) = [9] = 9 \)

\[ \therefore f(5) \neq f(9) \]

The differentiability of \( f \) in \( (5, 9) \) is checked as follows.

Let \( n \) be an integer such that \( n \in (5, 9) \).

The left hand limit of \( f \) at \( x = n \) is,

\[ \lim_{{h \to 0^+}} \frac{f(n+h)-f(n)}{h} = \lim_{{h \to 0^+}} \frac{[n+h]-[n]}{h} = \lim_{{h \to 0^+}} \frac{n+1-n}{h} = \lim_{{h \to 0^+}} \frac{1}{h} = \infty \]

The right hand limit of \( f \) at \( x = n \) is,

\[ \lim_{{h \to 0^-}} \frac{f(n+h)-f(n)}{h} = \lim_{{h \to 0^-}} \frac{[n+h]-[n]}{h} = \lim_{{h \to 0^-}} \frac{n-n}{h} = \lim_{{h \to 0^-}} 0 = 0 \]

Since the left and right hand limits of \( f \) at \( x = n \) are not equal, \( f \) is not differentiable at \( x = n \)
\[ f \text{ is not differentiable in } (5, 9). \]

It is observed that \( f \) does not satisfy all the conditions of the hypothesis of Rolle's Theorem. Hence, Rolle's Theorem is not applicable for \( f(x) = [x] \text{ for } x \in [5, 9] \).

\[(ii) \quad f(x) = [x] \text{ for } x \in [-2, 2]\]

It is evident that the given function \( f(x) \) is not continuous at every integral point. In particular, \( f(x) \) is not continuous at \( x = -2 \) and \( x = 2 \):

\[\Rightarrow f(x) \text{ is not continuous in } [-2, 2].\]

Also, \( f(-2) = [-2] = -2 \) and \( f(2) = [2] = 2 \):

\[\therefore f(-2) \neq f(2)\]

The differentiability of \( f \) in \( (-2, 2) \) is checked as follows.

Let \( n \) be an integer such that \( n \in (-2, 2) \).

The left hand limit of \( f \) at \( x = n \) is,

\[\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{[n+h]-[n]}{h} = \lim_{h \to 0} \frac{n+1-n}{h} = \lim_{h \to 0} \frac{1}{h} = \infty.\]

The right hand limit of \( f \) at \( x = n \) is,

\[\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{[n+h]-[n]}{h} = \lim_{h \to 0} \frac{n-n}{h} = \lim_{h \to 0} 0 = 0.\]

Since the left and right hand limits of \( f \) at \( x = n \) are not equal, \( f \) is not differentiable at \( x = n \):

\[\therefore f \text{ is not differentiable in } (-2, 2).\]

It is observed that \( f \) does not satisfy all the conditions of the hypothesis of Rolle's Theorem. Hence, Rolle's Theorem is not applicable for \( f(x) = [x] \text{ for } x \in [-2, 2] \).

\[(iii) \quad f(x) = x^2 - 1 \text{ for } x \in [1, 2]\]

It is evident that \( f \), being a polynomial function, is continuous in \([1, 2]\) and is differentiable in \((1, 2)\).

\[f(1) = 1^2 - 1 = 0\]
\[f(2) = 2^2 - 1 = 3\]
\[\therefore f(1) \neq f(2)\]

It is observed that \( f \) does not satisfy a condition of the hypothesis of Rolle's Theorem. Hence, Rolle's Theorem is not applicable for \( f(x) = x^2 - 1 \text{ for } x \in [1, 2] \).

Question
If \( f : [-5, 5] \to \mathbb{R} \) is a differentiable function and if \( f'(x) \) does not vanish anywhere, then prove that \( f(-5) \neq f(5) \).

Answer:
It is given that \( f : [-5, 5] \to \mathbb{R} \) is a differentiable function.
Since every differentiable function is a continuous function, we obtain
(a) \( f \) is continuous on \([-5, 5]\).
(b) \( f \) is differentiable on \((-5, 5)\).
Therefore, by the Mean Value Theorem, there exists \( c \in (-5, 5) \) such that
\[
 f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}
\]
\[
\Rightarrow 10 f'(c) = f(5) - f(-5)
\]
It is also given that \( f'(x) \) does not vanish anywhere.
\[
\therefore f'(c) \neq 0
\]
\[
\Rightarrow 10 f'(c) \neq 0
\]
\[
\Rightarrow f(5) - f(-5) \neq 0
\]
\[
\Rightarrow f(5) \neq f(-5)
\]
Hence, proved.

Question
Verify Mean Value Theorem, if \( f(x) = x^3 - 4x - 3 \) in the interval \([a, b]\), where \( a = 1 \) and \( b = 4 \).

**Answer:**

The given function is \( f(x) = x^3 - 4x - 3 \). 

\( f \) being a polynomial function, is continuous in \([1, 4]\) and is differentiable in \((1, 4)\) whose derivative is \( 2x - 4 \).

\[
\begin{align*}
 f(1) &= 1^3 - 4\times1 - 3 = -6, \\
 f(4) &= 4^3 - 4\times4 - 3 = -3 \\
 \Rightarrow f(b) - f(a) &= \frac{f(4) - f(1)}{4-1} = \frac{-3 - (-6)}{3} = \frac{3}{3} = 1
\end{align*}
\]

Mean Value Theorem states that there is a point \( c \in (1, 4) \) such that \( f'(c) = 1 \)

\[
 f'(c) = 1 \\
 \Rightarrow 3c^2 - 4 = 1 \\
 \Rightarrow c = \frac{5}{2}, \text{ where } c = \frac{5}{2} \in (1, 4)
\]

Hence, Mean Value Theorem is verified for the given function.

**Question**

Verify Mean Value Theorem, if \( f(x) = x^3 - 5x^2 - 3x \) in the interval \([a, b]\), where \( a = 1 \) and \( b = 3 \). Find all \( c \in (1, 3) \) for which \( f'(c) = 0 \)

**Answer:**

The given function \( f \) is \( f(x) = x^3 - 5x^2 - 3x \). 

\( f \) being a polynomial function, is continuous in \([1, 3]\) and is differentiable in \((1, 3)\) whose derivative is \( 3x^2 - 10x - 3 \).

\[
\begin{align*}
 f(1) &= 1^3 - 5\times1^2 - 3\times1 = -7, \\
 f(3) &= 3^3 - 5\times3^2 - 3\times3 = -27 \\
 \Rightarrow f(b) - f(a) &= \frac{f(3) - f(1)}{3-1} = \frac{-27 - (-7)}{3-1} = -10
\end{align*}
\]

Mean Value Theorem states that there exist a point \( c \in (1, 3) \) such that \( f'(c) = -10 \)

\[
 f'(c) = -10 \\
 \Rightarrow 3c^2 - 10c - 3 = 10 \\
 \Rightarrow 3c^2 - 10c + 7 = 0 \\
 \Rightarrow 3(c^2 - \frac{10}{3}c + \frac{7}{3}) = 0 \\
 \Rightarrow 3(c - \frac{5}{3})^2 = 0 \\
 \Rightarrow c = \frac{5}{3}
\]
\[ (c-1)(3c-7) = 0 \]
\[ \Rightarrow c = \frac{7}{3}, \text{ where } c = \frac{7}{3} \in (1, 3) \]

Hence, Mean Value Theorem is verified for the given function and \( c = \frac{7}{3} \in (1, 3) \) is the only point for which \( f'(c) = 0 \)

**Question**

Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

**Answer:**

Mean Value Theorem states that for a function \( f: [a, b] \rightarrow \mathbb{R} \), if

(a) \( f \) is continuous on \([a, b]\)

(b) \( f \) is differentiable on \((a, b)\)

then, there exists some \( c \in (a, b) \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i) \( f(x) = [x] \) for \( x \in [5, 9] \)

It is evident that the given function \( f(x) \) is not continuous at every integral point. In particular, \( f(x) \) is not continuous at \( x = 5 \) and \( x = 9 \)
\[ f(x) \text{ is not continuous in } [5, 9]. \]

The differentiability of \( f \) in (5, 9) is checked as follows.

Let \( n \) be an integer such that \( n \in (5, 9) \).

The left hand limit of \( f \) at \( x = n \) is,

\[
\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{(n+h) - n}{h} = \lim_{h \to 0} \frac{n+1-n}{h} = \lim_{h \to 0} \frac{-1}{h} = \infty
\]

The right hand limit of \( f \) at \( x = n \) is,

\[
\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{(n+h) - n}{h} = \lim_{h \to 0} \frac{n-n}{h} = \lim_{h \to 0} \frac{0}{h} = 0
\]

Since the left and right hand limits of \( f \) at \( x = n \) are not equal, \( f \) is not differentiable at \( x = n \)

\( \therefore f \) is not differentiable in (5, 9).

It is observed that \( f \) does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for \( f(x) = \lfloor x \rfloor \) for \( x \in [5, 9] \).

(ii) \( f(x) = \lfloor x \rfloor \) for \( x \in [-2, 2] \)

It is evident that the given function \( f(x) \) is not continuous at every integral point.

In particular, \( f(x) \) is not continuous at \( x = -2 \) and \( x = 2 \)

\[ \Rightarrow f(x) \text{ is not continuous in } [-2, 2]. \]

The differentiability of \( f \) in (-2, 2) is checked as follows.

Let \( n \) be an integer such that \( n \in (-2, 2) \).

The left hand limit of \( f \) at \( x = n \) is,

\[
\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{(n+h) - n}{h} = \lim_{h \to 0} \frac{n+1-n}{h} = \lim_{h \to 0} \frac{-1}{h} = \infty
\]

The right hand limit of \( f \) at \( x = n \) is,

\[
\lim_{h \to 0} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0} \frac{(n+h) - n}{h} = \lim_{h \to 0} \frac{n-n}{h} = \lim_{h \to 0} \frac{0}{h} = 0
\]

Since the left and right hand limits of \( f \) at \( x = n \) are not equal, \( f \) is not differentiable at \( x = n \)

\( \therefore f \) is not differentiable in (-2, 2).

It is observed that \( f \) does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for \( f(x) = \lfloor x \rfloor \) for \( x \in [-2, 2] \).

(iii) \( f(x) = x^2 - 1 \) for \( x \in [1, 2] \)

It is evident that \( f \), being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).
It is observed that $f$ satisfies all the conditions of the hypothesis of Mean Value Theorem. Hence, Mean Value Theorem is applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

It can be proved as follows.

\[
\begin{align*}
f(1) & = 1^2 - 1 = 0, \quad f(2) = 2^2 - 1 = 3 \\
\therefore \quad \frac{f(b) - f(a)}{b-a} & = \frac{f(2) - f(1)}{2-1} = \frac{3-0}{1} = 3 \\
f''(x) & = 2x \\
\therefore \quad f''(c) & = 3 \\
\Rightarrow \quad 2c & = 3 \\
\Rightarrow \quad c & = \frac{3}{2} = 1.5, \text{ where } 1.5 \in [1, 2]
\end{align*}
\]
Review of Graphs

The student must be very good at Graphs of Various kinds of functions; to do well in Continuity, Differentiability, Area, Volume problems. Some limit problems also require concepts of Functions and Graphs. The graphs will not be given in the Questions. In case of Area problems, the student has to draw the graphs quickly, largely to scale; get the intersection points, and then plan for a piece-wise strategy to integrate and find the area.

Let us review the various graphs

\(y = mx\) will be a straight line passing through the origin. Positive \(m\) will make the line move upwards as we move in positive \(x\) i.e. towards right.

![Graph of \(y = 2x\)](image)

This is graph of \(y = 2x\)  Don’t get foxed by the angle being almost 45° The scales in y-axis and x-axis are not same.

If we compare two graphs then it becomes more clear.

![Comparison of graphs](image)

In this figure also scales of x-axis and y-axis are not same. But \(y = 6x\) has to be steeper than \(y = 2x\)
This is $y = 3x$ and $y = -5x$ graphs. For $m = -5$ the line moves down.

For $y = mx + c$ the $c$ becomes the intercept in the y-axis.

So $y = 3x - 4$ will look like

If $c$ is a positive number then the intercept in $y$-axis will be on upper (positive) side.

Graphs of $y = 2x + 3$ and $y = 4x + 5$ will be
Again scales in x-axis and y-axis are different. But point made. See how the graphs pass through 3 and 5 respectively.

Nature of Curves, Types of Graphs, Shapes are explained / discussed at


Now let us see graphs of Quadratic functions

Graph of $y = x^2$ will be

In contrast graph of $y = -3x^2$ will be downwards
Graph of $y = (1/3) \ x^2$ will be flatter compared to $y = x^2$.
Similarly graph of $y = 10x^2$ will be narrow and steeper compared to $y = x^2$.

So see comparisons in a single image.
Similar things happen with power functions as well. Below we see fraction raised to power $x$.

Let us see the graph of $y = 2^x$.

plot $y = 2^x$
The graph of $y = 3^x$ will be steeper and is understood easily by comparison.

Now let us compare integer to the power $x$ and fraction to the power $x$.
What about comparing $y = 3^x$ and $y = -3^x$?

Spoon Feeding comparison of $y = 2^x$ and $y = -2^x$.
Graph of $y = 4x^2 + 3$ will be 3 units above $x$-axis. So will pass through $(0, 3)$. The parabola will look similar to $y = x^2$.

Let us learn more with graphs of $y = -5x^2 + 6$ and $y = 6x^2 - 7$.

Don’t quickly assume that the graphs are intersecting on $x$ axis. The roots are very close.

$5x^2 = 6 \Rightarrow x = \pm \sqrt{\frac{6}{5}} = \pm 1.095$

While $6x^2 = 7 \Rightarrow x = \pm \sqrt{\frac{7}{6}} = \pm 1.0801$
Concept of Shifting of graphs

The graph of $y = 3(x - 2)^2$ will be same as $y = 3x^2$ while shifted by 2 units towards right.

Similarly graph of $y = 4(x + 3)^2$ will be shifted by 3 units on left compared to $y = 4x^2$ which is through the origin.
In the above image see how the upper graph is shifted up by 1 due to +1

In the image below the graph is shifted down by -1

https://archive.org/details/AreaDefiniteIntegralIITJEE2005ShiftingParabolasLeftOrRight
The parabola that passes through (1,0) and (7,0) will be \((x - 1)(x - 7)\).

In simple words, the quadratic expression that has roots 1 and 7 is a parabola through 1 and 7.

So, the graph of \(y = (x - 1)(x - 7) = x^2 - 8x + 7\) is:

![Graph of \(y = x^2 - 8x + 7\)]

If a quadratic expression has roots -3, 5 then it will be a parabola passing through -3 and 5.

So, the graph of \(y = (x + 3)(x - 5) = x^2 - 2x - 15\) is:

![Graph of \(y = x^2 - 2x - 15\)]
If the Discriminant $D < 0$ i.e. $b^2 < 4ac$ then the whole parabola is above $x$-axis signifying imaginary roots. As the parabola does not intersect the $x$-axis at all. For $a > 0$ 

If $a$ is negative then the parabola will be downwards

So graph of $y = (x - 3)^2 + 5$ will be

Meaning minima will be at $x = 3$ so $x^2$ graph shifted right by 3 and added 5 so moved up by 5 units

So we can easily guess the graph of $y = - (x + 5)^2 - 7$ ....

It will be shifted left by 5 units. So maxima will be at $x = -5$ and 7 units below $x$ axis
The parabola is downwards because coeff of $x^2$ is -ve

Don’t use the idea of shift blindly! The graph of $y = e^{x-4}$ is not shifted by 4 units that of $y = e^x$

This is because $e^{(x-4)} = e^x / e^4$ means just divided by a value

Concept of Reflections

Guess the graph of $y = -e^x$
What about graph of $y = e^{-x}$ and $y = -e^{-x}$

Graph of $y = x \ln x$ (Ignore the Imaginary part graph)
Graph of $y = x \ln |x|$

How will the graph of $y = (\ln x)/x$ look like? (Ignore the Imaginary part)
What about graph of $y = \frac{\ln |x|}{x}$

IIT-JEE 1990 problem and Solution on Area, Tricky graph of $x \ln x$ is explained / Discussed at

https://archive.org/details/AreaDefiniteIntegralIITJEE1990TrickyGraphsOfXLnXAndLnXByX
Graph of Floor $x$, i.e., greatest integer function $x$, $y = \lfloor x \rfloor$

Recall $\lfloor -3.2 \rfloor$ is $-4$ the integer less than $-3.2$ while $\lfloor -3.99 \rfloor$ is also $-4$

What about graph of $y = -\lfloor x \rfloor$ (i.e., negative of Floor function)
Best way to learn is to “think” and try to plot it yourself, in rough.

There are many theorems related to “Floor or Greatest Integer functions”. Two theorems related to Floor function are discussed while solving a complicated Limit problem.

https://archive.org/details/VeryImportantTwoFloorTheoremsGreatestIntegerFunctionExplanationAndExample

Fraction $x$ can be defined as $x - \lfloor x \rfloor$ so graph of $y = \{ x \}$ will be

<table>
<thead>
<tr>
<th>$\lfloor 2.3 \rfloor$</th>
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<tr>
<td>$\lfloor 2.4 \rfloor$</td>
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<td>$\lfloor 4.5 \rfloor$</td>
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<tr>
<td>$\lfloor 4.6 \rfloor$</td>
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![Graph of $y = \{ x \}$](attachment:Graph.png)
There are infinite number of discontinuities.

Graph of $y = \ln(x)$  

Note: Log of negative number is imaginary as discussed in the complex number chapter.

Graph of $y = \ln|x|$ and $y = -\ln|x|$
Graph of $y = \sin x$ vs $y = \sin^{-1} x$

Not sure if the above graph communicates well. Imaginary part of the graph to be ignored / avoided as of this discussion.

$y = \sin^{-1} x$ means $x = \sin y$ The graph of which is drawn much easier.
I am sure this is much better

Graph of $y = \cos x$ vs $y = \cos^{-1} x$
Graph of $y = \sec x$ vs $y = \sec^{-1} x$

I guess we should see these graphs individually as these graphs are not commonly given in other text books.
Actually Cos x can be drawn in the gap to fit-in well

Graph of y = Cosec x

Y = Sin x has been fit into this
Graph of $y = \tan x$

Graph of $y = \tan^{-1} x$
Let us compare these a few more times, so that we can remember
Graph of $y = \cot^{-1} x$
An introduction to Periodic functions, Decision to Multiply or Divide is explained at

https://archive.org/details/PeriodicFunctionsAnIntroductionOfPeriodMultiplyOrDivide

- Graphs of modulus functions
Valentine's Day: \( y = |x| \pm \sqrt{4 - x^2} \)
Now let us see Horizontal Parabolas

Graph of \( y^2 = 4x \) is of the form \( y^2 = 4a \times x \)
Graphs of Cubic Equations ( \( y = x^3 \) ) and higher powers of \( x \)

Graph of \( y = x^3 \) is
A good student can learn a lot by thinking how the graph of negative of the same function will look.

\[ y = -x^2 \]
How will \( y = (x + 6)(x - 3)(x - 7) \) look like? \( x = -6, 3 \) and 7 will be roots. So the graph will pass through \((-6, 0), (3, 0)\) and \((7, 0)\).

If coeff of \( x \) cube is negative then the graph will be downwards for increasing \( x \). Also repeat roots can be there. Try to guess the graph of \( y = (5 - x)(2 - x)^2 \).

This will have roots at \( x = 5 \) and repeat roots \( \text{(Two roots)} \) at \( x = 2 \) so will touch \( x \) axis at \( x = 2 \).
Because of distorted scale this graph is not a good one. The graph is correct but student must be mature to understand the distorted scale effects.

The graph below is a better one from a different plotter.

The graph of $y = x^5$ or say $y = x^{11}$ will look very similar.
The difference is highlighted if the graphs are drawn together. All these graphs pass through \((1, 1)\) and \((-1, -1)\). While higher powered graph is flatter in between -1 to 1 and steeper after 1 or before -1.
Graph of \( y = x^2 (x + 3)(x - 3) = x^2(x^2 - 9) \)

\[ \text{plot} \quad y = x^2(x^2 - 9) \]

\( x = 0 \) will be repeat root due to \( x \) square. Also \( x = 3 \) and \( x = -3 \) will be the roots

Graph of \( y = -x^2(x^2 - 9) + 6 \)

\[ \text{plot} \quad y = 6 - x^2(x^2 - 9) \]

Now let us see graphs of Circles

Graph of \( x^2 + y^2 = R^2 \) will have the center at \( (0,0) \) and radius will be \( R \)

So graph of \( x^2 + y^2 = 36 \) is
Graph of \((x - 3)^2 + (y - 4)^2 = 25\) is

Center is at \((3, 4)\)

Area problems, Graphs of Line, Circle, Triangle Areas discussed and explained at

https://archive.org/details/AreaDefiniteIntegralLineCircleModulusTriangleNatureAndType
Some special graphs

\[ y = \frac{x}{x^2 + 1} \]

The graph becomes asymptotic to the x-axis as we move towards right or left very slowly.

\[ y = \frac{x^2}{x^2 + 1} \]

\[ \lim_{x \to \infty} \frac{x^2}{1 + x^2} = 1 \]

In this case the graph is asymptotic to 1 (\( y = \frac{x^2}{x^2 + 1} \))
Can you guess what will happen in case of \[ y = \frac{x^2}{x^3 + 2} \] ? Did you notice the discontinuity around negative cube root of 2?

plot \[ y = \frac{x^2}{x^3 + 2} \]
Can you guess what will happen in case of \( y = \frac{x}{x^3 + 4} \)? Understand the discontinuity around negative cubic root of 4.

\[
\text{plot } y = \frac{x}{x^3 + 4}
\]

Find all asymptotes and sketch the function

\[
\ell(x) = \frac{x^6 + 5}{x^2 + 3x + 1}
\]

\[
x^2 + 3x + 1 = 0
\]

\[-3 \leq x \leq 1\]

\[
x = \text{--------- (2 vertical asymptotes)}
\]

\[
\frac{(x^3/x^3) + (5/x^3)}{2} = \text{undefined (no horizontal asymptotes)}
\]

\[
\text{Undefined} = \frac{x - 3 + ((8x + 8)/(x^2 + 3x + 1))}{x^2 + 3x + 1 / x^3 + 0x^2 + 0x + 5}
\]

\[
x^3 + 3x^2 + x
\]

\[
\text{Undefined} = \frac{-3x^2 - x + 5}{-3x^2 - 9x - 3}
\]

\[
8x + 8
\]
Find all asymptotes and sketch the function

\[ g(x) = \frac{x^2}{x - 3} \]

\[ x - 3 = 0 \]
\[ x = 3 \text{ (one vertical asymptote)} \]

\[ y = \frac{x^2}{x^2 - 3x^2} = \text{undefined (no horizontal asymptotes)} \]

\[ x + \frac{(3x)/(x - 3))}{x - 3 / x^2 + 0x + 0} \]
\[ x^2 - 3x \]
\[ 3x \]

\[ y = x + \frac{3x}{x} = x + 3 \text{ (one oblique asymptote)} \]
Find all asymptotes and sketch the function

\[ \frac{x^3 - 4x^2 - 49x - 90}{2x^2 + 12x + 18} \]

\[ x = -3 \text{ (one vertical asymptote)} \]

\[ \frac{x^3/x^3 - 4x^2/x^3 - 49x/x^3 - 90/x^3}{2x^2/x^3 + 12x/x^3 + 18/x^3} = \text{undefined (no horizontal asymptotes)} \]

\[ 0.5x - 5 + \frac{(2x)/(2x^2 + 12x + 18))}{-10x^2 - 58x - 90} \]

\[ -10x^2 - 60x - 90 \]
\[ y = 0.5x - 5 + \frac{2x}{2x^2 + 12x + 18} \]
\[ = 0.5x - 5 + \frac{2x}{x^2 + 12} + \frac{18}{x^2} \]
\[ = 0.5x - 5 \] (one oblique asymptote)

Find all asymptotes and sketch the function

\[ h(x) = \frac{4x^5 - 6}{9x^5 + 7x^2} \]
\[ 9x^5 + 7x^2 = x^2(9x^3 + 7) = 0 \]
\[ x = (-7/9)^{1/3} \text{ or } x = 0 \] (two vertical asymptotes)

\[ y = \frac{4x^5 - 6}{9x^5 + 7x^2} \]
\[ = \frac{4}{9} \] (one horizontal asymptote)

There are no oblique asymptotes, as the degree of the numerator is not one greater than the degree of the denominator.
Find all asymptotes and sketch the function

\[
y = \frac{x^4 - 3x^3 + 5x^2 - 7x + 9}{x^5 - x^4 - x^3 + 3x^2 - 5x + 18}
\]

First, reduce the equation to \( y = 1/(x + 2) \)

\[
x + 2 = 0 \\
x = -2 \text{ (one vertical asymptote)}
\]

\[
y = \frac{1}{x} = 0 \text{ (one horizontal asymptote)} \\
\frac{x}{x} + \frac{2}{x}
\]

There are no oblique asymptotes, as the degree of the numerator is not one greater than the degree of the denominator.
Graphs of $y = x + 1/x$ and $y = x - 1/x$
Graph of 

$$y = \frac{1}{2} \left( e^{x^2} + e^{-x^2} \right)$$

So graph of 

$$y = \frac{1}{e^{x^2} + e^{-x^2}}$$
Spoon Feeding graph of \( y = \frac{1}{e^x + e^{-x}} \)

If area enclosed between two curves is needed; then the upper curve function minus the lower curve function needs to be integrated, between the two intersection points as limits.
We generally get questions with line intersecting a parabola kind ...

\[ y = x^2 - 2 \]

\[ y = -x^2 + 10 \]

\[ x = (y - 2)^2 \]
Draw the graphs of \( y = x^2 \) and \( y = 2x - x^2 \)
Draw the graph of the parabola $y = 4x - x^2$

Draw the graph of $f(x) = x^2 - 6x + 10$, the lines $x = 2$ and $x = 5$ and the $x$-axis
Draw the graph of \( x = 4 - y^2 \Rightarrow y^2 = 4 - x \)

\[
\begin{align*}
\text{plot } & \quad y^2 = 4 - x \\
\end{align*}
\]

Draw the graph of \( y = 4x + 16 \) and \( y = 2x^2 + 10 \)

\[
\begin{align*}
\text{Solving these two given equations we get the intersection points as } x = -1 \text{ and } x = 3 \text{ ( Quadratic equation } 2x^2 + 10 = 4x + 16 \Rightarrow 2x^2 - 4x - 6 = 0 \\
\Rightarrow x^2 - 2x - 3 = 0 \text{ Factorize and you get } x = -1 \text{ and } x = 3 )}
\end{align*}
\]
Draw the graphs of $x = y^2$ and $y = x^2$

![Graph of $x = y^2$ and $y = x^2$](image)

We can easily solve to see that the graphs intersect at (1, 1).

Draw the graphs of $y = 2x^2 + 10$, $y = 4x + 16$, $x = -2$, and $x = 5$

The regions in the graph needs to be plotted.
Draw the graphs of $4y^2 = 9x$ and $3x^2 = 16y$

Draw the graph of $y^2 = 6x + 4$
Draw the graph of $y = x^3 - 4x$
Graph of $y = 4 - x^2$

We need to know graphs of ellipse and problems related to those.
The line and the parabola intersect at $y = -2 \Rightarrow x = y + 1 = -1$ so $(-1, -2)$

and $y = 4 \Rightarrow x = y + 1 = 5$ so $(5, 4)$
the function becomes \( y = \pm \sqrt{2x+6} \)

Graph of \( x = y^3 - 3 \) and \( x = (y-2)^2 \).

The intersection points are \( y = -1 \) and \( y = 3 \).
See the graphs
Spoon Feed

Draw the graphs of $x = 2$ and $y^2 = 8x$
Graph of general \( x = a \) and \( y^2 = 4ax \)

Graph of ellipse \( 4x^2 + 9y^2 = 36 \)
Graphs of \( y^2 = x \) and the line \( y = x \)

Spoon Feeding Graph of \( y^2 = 4x \) and \( y = 2x \)

We see the intersection point is \((1, 2)\).
Graphs of ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) and the line \( \frac{x}{a} + \frac{y}{b} = 1 \)

Assuming \( a > b \)

Graph of curve \( y = 2\sqrt{1 - x^2} \)
Graph of Circle. The equation of the circle will be \( x^2 + y^2 = a^2 \) so \( y = \sqrt{a^2 - x^2} \)

Graphs of \( 2y = 5x + 7 \), \( x = 2 \), and \( x = 8 \)

Draw the triangle. The vertices being A (2,1), B (3,4), C (5,2)
Draw the graphs \( y = 2x + 1 \) (line A), \( y = 3x + 1 \) (line B), \( y = 4 \) (line AC)
Draw graphs of \( y^2 \leq 8x \), \( x^2 + y^2 \leq 9 \)

The graph will be

Graphs of \( x^2 + y^2 = 16 \), and \( y^2 = 6x \)

The graph will be
Graphs of $x^2 + y^2 = 4$, and $(x - 2)^2 + y^2 = 4$

Equation (1) is a circle with centre O at the origin and radius 2. Equation (2) is a circle with centre C (2, 0) and radius 2. Solving equations (1) and (2), we have

$(x - 2)^2 + y^2 = x^2 + y^2$

Or $x^2 - 4x + 4 + y^2 = x^2 + y^2$

Or $x = 1$ which gives $y = \pm \sqrt{3}$

Thus, the points of intersection of the given circles are $A (1, \sqrt{3})$ and $A' (1, -\sqrt{3})$
Graphs of $y^2 = x$ and $x + y = 2$

Equation (1) represents a parabola with vertex at origin and its axis as $x$-axis, equation (2) represents a line passing through $(2,0)$ and $(0,2)$. Points of intersection of line and parabola are $(1,1)$ and $(4,-2)$.

A rough sketch of curves is as below:

\[ y^2 = x \]

\[ x + y = 2 \]

Graphs of $x = -2$, $x = 3$, $x$-axis ($y = 0$), and $y = 1 + |x + 1|$.

The straight lines for the mod function will flip around $x = -1$.
So, equation (1) is a straight line that passes through (0,2) and (-1,1). Equation (2) is a line passing through (-1,1) and (-2,2) and it is enclosed by line \( x = 2 \) and \( x = 3 \) which are lines parallel to y-axis and pass through (2,0) and (3,0) respectively. \( y = 0 \) is x-axis. So, a rough sketch of the curves is given as:--

Shaded region represents the required area.

Draw \( 0 < x < 1 \) for \( y = | x - 5 | \)

The graph of the modulus function will flip around \( x = 5 \)
Graphs of Hyperbolas

<table>
<thead>
<tr>
<th></th>
<th>$x$-axis</th>
<th>$y$-axis</th>
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</thead>
<tbody>
<tr>
<td><strong>Equation</strong></td>
<td>$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$</td>
<td>$-\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$</td>
</tr>
<tr>
<td><strong>Center</strong></td>
<td>$(h, k)$</td>
<td>$(h, k)$</td>
</tr>
<tr>
<td><strong>Semi-transverse axis</strong></td>
<td>$a$</td>
<td>$a$</td>
</tr>
<tr>
<td><strong>Semi-conjugate axis</strong></td>
<td>$b$</td>
<td>$b$</td>
</tr>
<tr>
<td><strong>Vertices</strong></td>
<td>$V(h \pm a, k)$</td>
<td>$V(h, k \pm a)$</td>
</tr>
<tr>
<td><strong>Foci</strong></td>
<td>$F(h \pm ae, k)$</td>
<td>$F(h, k \pm ae)$</td>
</tr>
<tr>
<td><strong>Directrices</strong></td>
<td>$x = h \pm a/e$</td>
<td>$y = k \pm a/e$</td>
</tr>
<tr>
<td><strong>Asymptotes</strong></td>
<td>$bx \pm ay = (b(h \pm a) = 0$</td>
<td>$ax \pm by = (a(h \pm b)k = 0$</td>
</tr>
<tr>
<td><strong>Focal chord length</strong></td>
<td>$2b^2/a$</td>
<td>$2b^2/a$</td>
</tr>
<tr>
<td><strong>Eccentricity</strong></td>
<td>$e = \frac{\sqrt{a^2+b^2}}{a} &gt; 1$</td>
<td>$e = \frac{\sqrt{a^2+b^2}}{a} &gt; 1$</td>
</tr>
</tbody>
</table>

For both horizontal and vertical hyperbolas:

**skyes of asymptotes** = $\frac{b}{a}$
Rectangular Hyperbolas (where the eccentricity = \sqrt{2} \ (x^2 - y^2 = 1) and (xy = 1) type
\[ y (x - 2) = 3x + 10 \]
\[ \Rightarrow \quad y = \frac{3x + 10}{x - 2} \]

A rough sketch of the curves is given below:

---

Draw \( y^2 = 4x \) and \( x^2 = 4y \)
Draw abstract graph of \( y^2 = 4ax \) and \( x^2 = 4by \)

Equation (1) represents a parabola with vertex \( (0,0) \) and axis as \( x\)-axis, equation (2) represents a parabola with vertex \( (0,0) \) and axis as \( y\)-axis, points of intersection of parabolas are \( (0,0) \) and \( \left( 4a \frac{1}{2}, b \frac{2}{3}, 4a \frac{2}{3}, b \frac{1}{3} \right) \)

A rough sketch is given as:-

![Graph of y^2 = 4ax and x^2 = 4by](image-url)
Draw graphs of $x^2 + y^2 = 4$ and $x = \sqrt{3}y$.

\[ x^2 + y^2 = 4 \]
\[ y = \frac{x}{\sqrt{3}} \]
Draw Graphs of $y = |x - 1|$ and $y = -|x - 1| + 1$

Draw $x^2 + y^2 = 16a^2$ and $y^2 = 6ax$

Equation (1) represents a circle with centre $(0,0)$ and meets axes $(\pm 4a, 0), (0, \pm 4a)$.
Equation (2) represents a parabola with vertex $(0,0)$ and axis as x-axis. Points of intersection of circle and parabola are $\{2a, 2\sqrt{3}a\}, \{2a, -2\sqrt{3}a\}$. 
Draw \( x^2 + y^2 = 8x \) and \( (x - 4)^2 + y^2 = 16 \) and \( y^2 = 4x \)

Equation (1) represents a circle with centre \((4,0)\) and meets axes at \((0,0)\) and \((8,0)\).
Equation (2) represent a parabola with vertex \((0,0)\) and axis as \(x\)-axis. They intersect at \((4,-4)\) and \((4,4)\).

A rough sketch of the curves is as under:-

Shaded region is the required region
A rough sketch of curves is given as:

Region \( AOCA \) is sliced into rectangles with area \( (y_1 - y_2) \Delta x \). It slides from \( x = 0 \) to \( x = \sqrt{3} \), so

Graph of \( y = 2x^2 \) and \( y = x^2 + 4 \)

Equation (1) represents a parabola with vertex \((0,0)\) and axis as y-axis. Equation (2) represents a parabola with vertex \((0,4)\) and axis as y-axis. Points of intersection of parabolas are \((2,8)\) and \((-2,8)\).

A rough sketch of curves is given as:

Region \( AOCA \) is sliced into rectangles with area \( (y_1 - y_2) \Delta x \). And it slides from \( x = 0 \) to \( x = 2 \)
Graphs of $x = 0$, $x = 2$, $y = 2^x$, $y = 2x - x^2$

$\Rightarrow \quad y = -\left\{x^2 - 2x + 1 - 1\right\}$

$\Rightarrow \quad y = -\left[(x - 1)^2 - 1\right]$ 

$\Rightarrow \quad y = -(x - 1)^2 + 1$

$\Rightarrow \quad -(y - 1) = (x - 1)^2 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{--- (2)}$

Equation (2) represents a downward parabola with axis parallel to y-axis and vertex at $(1, -1)$. Table for equation (1) is

Graphs of $3x^2 + 5y = 32$ and $y = |x - 2|$ 

$3x^2 = 5 \left(y - \frac{32}{5}\right) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{--- (1)}$
\[ y = |x - 2| \]

\[ \Rightarrow y = \begin{cases} 
- (x - 2), & \text{if } x - 2 < 1 \\
(x - 2), & \text{if } x - 2 \geq 1 
\end{cases} \]

\[ \Rightarrow y = \begin{cases} 
2 - x, & \text{if } x < 2 \\
x - 2, & \text{if } x \geq 2 
\end{cases} \]  \hspace{1cm} (2)

Equation (1) represents a downward parabola with vertex \( \left(0, \frac{32}{5}\right) \) and equation (2) represents lines. A rough sketch of curves is given as:

Graphs of y-axis (i.e. \( x = 0 \)), and \( 4y = |4 - x^2| \)
Graphs of $x = 0, y = 1, y = 4, \text{ and } y = 4x^2$
Equation (1) represents a parabola with vertex \( (0,0) \) and axis as \( y \)-axis. \( x = 0 \) is \( y \)-axis and \( y = 1 \), \( y = 4 \) are lines parallel to \( x \)-axis passing through \( (0,1) \) and \( (0,4) \), respectively. A rough sketch of the curves is given as:

\[
\begin{align*}
\text{Graphs of } y^2 &= 2x + 1, \quad (1) \\
\text{and } x - y &= 1, \quad (2)
\end{align*}
\]

Equation (1) is a parabola with vertex \( \left(-\frac{1}{2}, 0\right) \) and passes through \( (0,1) \), \( (0,-1) \).
Equation (2) is a line passing through \( (1,0) \) and \( (0,-1) \). Points of intersection of parabola and line are \( (3,2) \) and \( (0,-1) \).
A rough sketch of the curves is given as:

![Diagram of a shaded region between curves]

Shaded region represents the required area. It is sliced in rectangles of area \( (x_1 - x_2) \Delta y \).

It slides from \( y = -1 \) to \( y = 3 \), so
Graphs of \( y = x - 1 \), \(- (1)\) and \(( y - 1 )^2 = 4 ( x + 1 )\)

Equation \( (1) \) represents a line passing through \((1,0)\) and \((0,-1)\). Equation \( (2) \) represents a parabola with vertex \((-1,1)\) passing through \((0,3), (0,-1), \left( -\frac{3}{4}, 0 \right)\). Their points of intersection \((0,-1)\) and \((8,7)\).

A rough sketch of curves is given as:

![Graph of curves](image)

Draw graphs of

\[
\begin{align*}
y &= 6x - x^2 \\
\Rightarrow &\quad -y = x^2 - 6x \\
\Rightarrow &\quad -y = x^2 - 6x + 9 - 9 \\
\Rightarrow &\quad -(y - 9) = (x - 3)^2 (1)
\end{align*}
\]

And

\[
\begin{align*}
y &= x^2 - 2x \\
y + 1 &= x^2 - 2x + 1 \\
(y + 1) &= (x - 1)^2 (2)
\end{align*}
\]
Equation (1) represents a parabola with vertex \((3,9)\) and downward. Equation (2) represents a parabola with vertex \((1,-1)\) and upward. Points of intersection of parabolas are \((0,0)\) and \((4,8)\). A rough sketch of the curves is given as:-

Graphs of \(y = x^2\), and \(y = |x|\)

The given area is symmetrical about \(y\)-axis.

\[
\therefore \text{Area } \triangle ABCD = \text{Area } \triangle MNO
\]
Graphs of $y = 2 - x^2$ -- (1) and $y + x = 0$ -- (2)

Equation (1) represents a parabola with vertex $(0, 2)$ and downward, meets axes at $\pm \sqrt{2}, 0$.
Equation (2) represents a line passing through $(0, 0)$ and $(2, -2)$. The points of intersection of line and parabola are $(2, -2)$ and $(-1, 1)$.
A rough sketch of curves is as follows:

![Graph of y = 2 - x^2 and y + x = 0]

Shaded region is sliced into rectangles with area $= (y_1 - y_2)\Delta x$. It slides from $x = -1$ to $x = 2$, so

-
Graphs of \( x^2 = 4y \) \( \cdots (1) \) and \( x = 4y - 2 \) \( \cdots (2) \)

Shaded area OBAO.

Let A and B be the points of intersection of the line and parabola.

Coordinates of point A are \((-1, \frac{1}{4})\).

Graphs of

\[
\begin{align*}
y &= 4x - x^2 \\
\Rightarrow & \quad -y = x^2 - 4x + 4 - 4 \\
\Rightarrow & \quad -y + 4 = (x - 2)^2 \\
\Rightarrow & \quad -(y - 4) = (x - 2)^2 \quad (1)
\end{align*}
\]

And

\[
\begin{align*}
\text{and } y &= x^2 - x \\
\left(y + \frac{1}{4}\right) &= \left(x - \frac{1}{2}\right)^2 \quad (2)
\end{align*}
\]
Equation (1) represents a parabola downward with vertex at \((2, 4)\) and meets axes at \((4, 0), (0, 0)\). Equation (2) represents a parabola upward whose vertex is \(\left(\frac{1}{2}, -\frac{1}{4}\right)\) and meets axes at \((1, 0), (0, 0)\). Points of intersection of parabolas are \((0, 0)\) and \(\left(\frac{5}{2}, \frac{15}{4}\right)\).

A rough sketch of the curves is as under: -
Graphs of $x = 0$, $x = 1$ and $y = x \quad \text{(1)}$ and $y = x^2 + 2 \quad \text{(2)}$

Equation (1) is a line passing through $(2,2)$ and $(0,0)$. Equation (2) is a parabola upward with vertex at $(0,2)$. A rough sketch of curves is as under:
Graphs of \( x = y^2 \) \( \text{-- (1)} \) and \( x = 3 - 2y^2 \) \( \text{-- (2)} \)

Equation (1) represents an upward parabola with vertex \((0,0)\) and axis \(-y\). Equation (2) represents a parabola with vertex \((3,0)\) and axis as \(x\)-axis. They intersect at \((1, -1)\) and \((1, 1)\). A rough sketch of the curves is as under:

![Graph of parabolas](image)

Graphs of \( y = 4x - x^2 \) \( \text{-- (1)} \) \( y = x^2 - x \) \( \text{-- (2)} \)

Given curves are
\[
y = 4x - x^2
\]
\[
\Rightarrow -\left(y - 4\right) = \left(x - 2\right)^2
\]
and
\[
y = x^2 - x
\]
\[
\Rightarrow \left(y + \frac{1}{4}\right)^2 = \left(x - \frac{1}{2}\right)^2
\]
Equation (1) represents a parabola downward with vertex at \( (2, 4) \) and meets axes at \( (4, 0), (0, 0) \). Equation (2) represents a parabola upward whose vertex is \( \left( \frac{1}{2}, -\frac{1}{4} \right) \)
and meets axes at \( (1, 0), (0, 0) \) and \( \left( \frac{5}{2}, \frac{15}{4} \right) \). A rough sketch of the curves is as under:

Graphs of \( y = |x - 1| \) -- (1) and \( y = 3 - |x| \) -- (2)

\[
\begin{align*}
\text{Area of the region above } x\text{-axis} & \\
\text{Graphs of } y = |x - 1| \quad \text{and } y = 3 - |x| \quad \text{-- (2)}
\end{align*}
\]

\[
\begin{align*}
y &= |x - 1| \\
\Rightarrow y &= \begin{cases} 1 - x, & \text{if } x < 1 \\ x - 1, & \text{if } x \geq 1 \end{cases} & \text{-- (1)} \\
\text{and } y &= 3 - |x| \\
\Rightarrow y &= \begin{cases} 3 + x, & \text{if } x < 0 \\ 3 - x, & \text{if } x \geq 0 \end{cases} & \text{-- (2)}
\end{align*}
\]
Drawing the rough sketch of lines \(1\), \(2\), \(3\) and \(4\) as under:

\[ y = x \]

\[ y = 3 + x \]

\[ y = -1 \cdot 2x \]

\[ y = 3 - x \]

Shaded region is the required area
Graphs of $y = x|x| - (1)$ and $x = -1$ and $x = 1$.

Required area $= \int_{-1}^{1} y \, dx$

$= \int_{-1}^{0} x|x| \, dx + \int_{0}^{1} x^2 \, dx$

$= \left[ \frac{x^3}{3} \right]_{-1}^{0} + \left[ \frac{x^3}{3} \right]_{0}^{1}$

$= -\left( \frac{1}{3} \right) + \frac{1}{3}$

$= \frac{2}{3}$ sq. units
Graphs of $x^2 + y^2 = 16$ -- (1) and $y^2 = 6x$ -- (2)

A function is a relation for which there is only one value of $y$ corresponding to any value of $x$. We sometimes write $y = f(x)$, which is notation meaning ‘$y$ is a function of $x$’.

Some very common mathematical constructions are not functions. For example, consider the relation $x^2 + y^2 = 4$ because multiple values can satisfy the equation. If put $y = 0$, then for $x = 2$ and $x = -2$ both the expression is 4.

There is a simple test to check if a relation is a function, by looking at its graph. This test is called the vertical line test. If it is possible to draw any vertical line (a line of constant $x$) which crosses the graph of the relation more than once, then the relation is not a function. If more than one intersection point exists, then the intersections correspond to multiple values of $y$ for a single value of $x$. 
An inverse function is a function which “does the reverse” of a given function. More formally, if \( f \) is a function with domain \( X \), then \( f^{-1} \) is its inverse function if and only if for every \( x \in X \) we have

\[
f^{-1}(f(x)) = x
\]

A simple way to think about this is that a function, say \( y = f(x) \), gives you a \( y \)-value if you substitute an \( x \)-value into \( f(x) \). The inverse function tells you tells you which \( x \)-value was used to get a particular \( y \)-value when you substitute the \( y \)-value into \( f^{-1}(x) \).

If \( f(x) = 3x + 2 \) then find \( f^{-1}(x) \)

Solution:
Put \( y = 3x + 2 \) and solve for \( x \) => \( y - 2 = 3x \) or \( x = (y - 2)/3 \)
Now exchange \( x \) and \( y \)
So \( f^{-1}(x) = (x - 2)/3 \)

We say that a function \( g(x) \) is periodic if there is a positive or negative number \( T \) for which \( g(x + T) = g(x) \) for all \( x \). We call \( T \) a period of \( g(x) \).

A periodic function has many periods.

Since the graph of \( g \) repeats after \( x \) increases by \( T \), it also repeats after \( x \) increases by \( 2T \), or \( -3T \), or any integer multiple (positive or negative) of \( T \). This means that a periodic function always has many periods. (That’s why the definition refers to “a period” rather than “the period.”)

The period of a periodic function is its smallest positive period. It is the size of a single cycle.

If the function \( g(x) \) is periodic, then its frequency is the number of cycles per unit \( x \).

In general, if \( f \) is the frequency of a periodic function \( g(x) \) and \( T \) is its period, then we have

\[
f = \frac{1}{T} \quad \text{and} \quad T = \frac{1}{f}
\]
if the period is measured in seconds, then the frequency is measured in cycles per second.

The term Hertz is a special unit used to measure time frequencies; it equals one cycle per second. Hertz is abbreviated Hz; thus a kilohertz (kHz) and a megahertz (MHz) are 1,000 and 1,000,000 cycles per second, respectively. This unit is commonly used to describe sound, light, radio, and television waves.

Circular functions. While there are innumerable examples of periodic functions, two in particular are considered basic: the sine and the cosine. They are called circular functions
because they are defined by means of a circle. To be specific, take the circle of radius 1 centered at the origin in the x, y-plane. Given any real number t, measure a distance of t units around the circumference of the circle.

Start on the positive x-axis, and measure counterclockwise if t is positive, clockwise if t is negative. The coordinates of the point you reach this way are, by definition, the cosine and the sine functions of t, respectively:

\[ x = \cos(t) \]
\[ y = \sin(t) \]

The whole circumference of the circle measures 2π units. Therefore, if we add 2π units to the t units we have already measured, we will arrive back at the same point on the circle. That is, we get to the same point on the circle by measuring either t or t + 2π units around the circumference. We can describe the coordinates of this point two ways:

\[ (\cos(t), \sin(t)) \text{ or } (\cos(t + 2\pi), \sin(t + 2\pi)) \]

Thus

\[ \cos(t + 2\pi) = \cos(t) \quad \sin(t + 2\pi) = \sin(t) \]

so \( \cos(t) \) and \( \sin(t) \) are both periodic, and they have the same period, 2π.

Here are their graphs. By reading their slopes we can see \( (\sin t)' = \cos t \) and \( (\cos t)' = -\sin t \)

While graph of \( y = \sin(4t) \) will be
Their scales are identical, so it is clear that the frequency of \( \sin(4t) \) is four times the frequency of \( \sin(t) \). The general pattern is described in the following table.

<table>
<thead>
<tr>
<th>function</th>
<th>period</th>
<th>frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin(t) )</td>
<td>( \cos(t) )</td>
<td>( 2\pi )</td>
</tr>
<tr>
<td>( \sin(4t) )</td>
<td>( \cos(4t) )</td>
<td>( 2\pi/4 )</td>
</tr>
<tr>
<td>( \sin(bt) )</td>
<td>( \cos(bt) )</td>
<td>( 2\pi/b )</td>
</tr>
</tbody>
</table>

Notice that it is the frequency—not the period—that is increased by a factor of \( b \) when we multiply the input variable by \( b \).

Constructing a circular function with a given frequency

By using the information in the table, we can construct functions with any period or frequency whatsoever. For instance, suppose we wanted a cosine function \( x = \cos(bt) \) with a frequency of 5 cycles per unit \( t \). This means

\[
5 = \text{frequency} = \frac{b}{2\pi}
\]

which implies that we should set \( b = 10\pi \) and \( x = \cos(10\pi t) \). 

-
Domain and Range of functions (graphs)

In simple words - Given a function, the values of x that are allowed to be supplied to the function, so that all terms remain real, is the domain. So Domain is the values for x as it varies.

Range are the values $y = f(x)$ takes, keeping the function as Real. So Range is the way y varies.

Consider the parabola $y = f(x) = 3x^2 + 4$

The graph will be

We see that $y$ is minimum 5 or more. So Range is $[5, \infty)$ while Domain is $(-\infty, \infty)$ or $]-\infty, \infty[$.

Note: the round brackets ( or ) means open bracket. Meaning close to that value but not equal to.

$\infty$ is always written with open bracket “(“) “ as we can have a very high value, but exactly infinity is undefined. Other way of writing the same thing is $5 \leq y < \infty$

Thus the $\leq$ sign is giving the symbol “[ “

In some statement if we get $6 < y < 9$ then same thing will be written as $(6, 9)$

But $1 \leq y \leq 8$ will be $[1, 8]$.

For the above problem for all values (positive, negative, zero) of x the graph exists. So $-\infty < x < \infty$ watch the $\leq$ sign is not being used for infinity.

Now some Mathematicians take pleasure in writing the same thing as $]-\infty, \infty[$ as this is easier to print.
Spoon Feeding: Find Domain and Range for \( x = 6y^2 + 7 \) and write the answer in all methods.

Graph will be

![Graph image](image)

So Domain is: \( 7 \leq x \) or \( 7 \leq x < \infty \) or \( [7, \infty) \) or \([7, \infty[\)

And Range is: \( ]-\infty, \infty[ \) or \((-\infty, \infty) \) or \(-\infty < y < \infty\)

Spoon Feeding: One of the most favorite questions by Math teachers, of standard 11 is to ask Domain and Range of \( y = f(x) = \sqrt{\frac{9-x}{x-1}} \)

Let me try to solve this without drawing the graph. If the student can guess the graph, of course it becomes very easy to solve.

If \( x \) becomes more than 9 then Numerator\( (N) \) becomes negative while Denominator\( (D) \) remains positive. As imaginary value of \( y \) is not allowed, \( x \) has to be less than or equal to 9 so \( x \leq 9 \)

\( N \) can be zero, but \( D \) is not allowed to be zero as dividing by 0 is not defined. So \( x = 1 \) is not allowed. If \( x < 1 \) then the \( N \) remain +ve but \( D \) becomes -ve. As \( y \) cannot be imaginary, \( 1 < x \), meaning \( x \) has to be greater than 1. But \( x \) can be arbitrarily close to 1 say \( 1 + \delta \) where \( \delta \) is very small positive number. In that case \( D \) becomes \( \delta \) while \( N = 9 - (1 + \delta) = 8 - \delta \)

\( (8 - \delta) / \delta \) tends to \( \infty \) So Range for \( y \) will go upwards to infinity. But what will be the least value?

We already saw that \( N \) can be zero \((0)\) but not negative. So Range will be from 0 to infinity.

Now let us see the graph.
Let us write the Solutions

Domain : \(1 \leq x \leq 9\) or \((1, 9]\) or \([1, 9]\) \quad \text{Happy?}

Range : \(0 \leq y < \infty\) or \([0, \infty)\) or \([0, \infty[\)

Spoon Feeding : Find Domain and Range of \(y = f(x) = \sqrt{\frac{9-x}{x-1}}\)

If \(x\) is in between 1 and 8, say 6 then \(N\) is +ve while \(D\) is -ve, which is not allowed.

If \(x\) is greater than 8 \((8 < x)\) then both \(N\) and \(D\) are negative so fine for us, as \(y\) the function is real. If \(x\) is less than 1 say 0 or -100 then also \(y\) is real. As both \(N\), and \(D\) is positive.

We need to analyze the function when \(x\) is very close to 1 i.e. \(x = 1 - \delta\) where \(\delta\) is very small positive number. \(D\) becomes \(1 - (1 - \delta) = \delta\)

Positive \(D\) divided by \(\delta\) will tend to infinity.

Now we can write the Domain : \(8 \leq x\) or \(x < 1\)

While Range : \(0 \leq y < \infty\) or \([0, \infty)\) or \([0, \infty[\)

Let us confirm by drawing the graph. (Though in the exam you have to just guess the graph)
To recall standard integrals

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$\int f(x),dx$</th>
<th>$f(x)$</th>
<th>$\int f(x),dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$\frac{x^{n+1}}{n+1}$ $(n \neq -1)$</td>
<td>$[g(x)]^n$ $g'(x)$</td>
<td>$\frac{</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln</td>
<td>x</td>
<td>$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x$</td>
<td>$a^x$</td>
<td>$\frac{a^x}{\ln a}$ $(a &gt; 0)$</td>
</tr>
<tr>
<td>$\sin x$</td>
<td>$- \cos x$</td>
<td>$\sinh x$</td>
<td>$\cosh x$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$\sin x$</td>
<td>$\cosh x$</td>
<td>$\sinh x$</td>
</tr>
<tr>
<td>$\tan x$</td>
<td>$- \ln</td>
<td>\cos x</td>
<td>$</td>
</tr>
<tr>
<td>$\csc x$</td>
<td>$\ln</td>
<td>\tan x</td>
<td>$</td>
</tr>
<tr>
<td>$\sec x$</td>
<td>$\ln</td>
<td>\sec x + \tan x</td>
<td>$</td>
</tr>
<tr>
<td>$\sec^2 x$</td>
<td>$\tan x$</td>
<td>$\sec^2 x$</td>
<td>$\tanh x$</td>
</tr>
<tr>
<td>$\cot x$</td>
<td>$\ln</td>
<td>\sin x</td>
<td>$</td>
</tr>
<tr>
<td>$\sin^2 x$</td>
<td>$\frac{x}{2} - \frac{\sin 2x}{4}$</td>
<td>$\sinh^2 x$</td>
<td>$\sinh \frac{2x}{4} - \frac{x}{2}$</td>
</tr>
<tr>
<td>$\cos^2 x$</td>
<td>$\frac{x}{2} + \frac{\sin 2x}{4}$</td>
<td>$\cosh^2 x$</td>
<td>$\sinh \frac{2x}{4} + \frac{x}{2}$</td>
</tr>
</tbody>
</table>
Some series Expansions -

\[
\frac{\pi}{2} = \left( \frac{2}{1} \cdot \frac{2}{3} \right) \left( \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{8}{9} \right) \ldots
\]

\[
\pi = 4 \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \right)
\]

\[
\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots
\]

\[
\pi = \sqrt{8 \left( 1 - \frac{1}{3} \cdot \frac{1}{3} - \frac{1}{5} \cdot \frac{1}{5} - \frac{1}{7} \cdot \frac{1}{7} + \ldots \right)}
\]

\[
\frac{x^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

\[
\int_0^\infty \log x \, dx = -\frac{\pi}{2} \log 2 = \frac{\pi}{2} \log \frac{1}{2}
\]
Solve a series problem

\[
\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \text{ up to } \infty \text{ is } \frac{\pi^2}{8}.
\]

\[
\begin{align*}
\text{(a)} & \quad \frac{\pi^2}{4} & \quad \text{(b)} & \quad \frac{\pi^2}{6} & \quad \text{(c)} & \quad \frac{\pi^2}{8} & \quad \text{(d)} & \quad \frac{\pi^2}{12} \\
\end{align*}
\]

**Solution** We have

\[
\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \text{ up to } \infty
\]

\[
= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots \text{ up to } \infty
\]

\[
= \frac{\pi^2}{6} - \frac{1}{4} \left( \frac{\pi^2}{6} \right) = \frac{\pi^2}{8}
\]

\[
1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots = \frac{\pi^2}{12}
\]

\[
\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{24}
\]

\[
\sin \sqrt{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \cdots
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}
\]

\[
\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}
\]

\[
\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!}
\]

\[
\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad (-1 \leq x < 1)
\]
\[
\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots + \frac{2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!} + \cdots \quad |x| < \frac{\pi}{2}
\]

\[
\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots + \frac{B_{2n}x^{2n}}{(2n)!} + \cdots \quad |x| < \frac{\pi}{2}
\]

\[
\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \cdots + \frac{2(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!} + \cdots \quad 0 < |x| < \pi
\]

\[
\cot x = \frac{1}{x} - \frac{x^3}{3} - \frac{2x^5}{45} - \frac{2x^7}{945} - \frac{2^{2n}B_{2n}x^{2n-1}}{(2n)!} - \cdots \quad 0 < |x| < \pi
\]

\[
\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots
\]

\[
\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{4x^6}{45} + \cdots
\]

\[
\log (\cos x) = -\frac{x^2}{2} - \frac{2x^4}{4} - \cdots
\]

\[
\log (1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots
\]
\[
\sin^{-1} x = x + \frac{1}{2} x^3 + \frac{1.3}{2 \cdot 4} x^5 + \frac{1.3 \cdot 5}{2 \cdot 4 \cdot 6} x^7 + \cdots \quad |x| < 1
\]

\[
\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x
\]

\[
= \frac{\pi}{2} \left( x + \frac{1}{2} x^3 + \frac{1.3}{2 \cdot 4} x^5 + \frac{1.3 \cdot 5}{2 \cdot 4 \cdot 6} x^7 + \cdots \right) \quad |x| < 1
\]

\[
\tan^{-1} x = \begin{cases} 
\frac{\pi}{2} + \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots & \text{if } x > 1 \\
\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots & \text{if } x < 1 
\end{cases}
\]

\[
\sec^{-1} x = \cos^{-1} \left( \frac{1}{x} \right)
\]

\[
= \frac{\pi}{2} - \left( \frac{1}{x} + \frac{1}{2 \cdot 3x^3} + \frac{1.3}{2 \cdot 4 \cdot 3x^5} + \frac{1.3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7x^7} + \cdots \right) \quad |x| > 1
\]

\[
\csc^{-1} x = \sin^{-1} \left( \frac{1}{x} \right)
\]

\[
= \frac{1}{x} + \frac{1}{2 \cdot 3x^3} + \frac{1.3}{2 \cdot 4 \cdot 3x^5} + \frac{1.3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7x^7} + \cdots \quad |x| > 1
\]

\[
\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x
\]

\[
= \begin{cases} 
\frac{\pi}{2} - \left( x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right) & \text{if } |x| < 1 \\
px + \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots & \text{if } p = 0 \text{ if } x \geq 1 \\
px + \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots & \text{if } p = 1 \text{ if } x \leq 1
\end{cases}
\]
\[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

\[ \ln x = 2 \left[ \frac{x-1}{x+1} + \frac{1}{3} \left( \frac{x-1}{x+1} \right)^3 + \frac{1}{5} \left( \frac{x-1}{x+1} \right)^5 + \ldots \right] \]

\[ = 2 \sum_{n=1}^{\infty} \frac{1}{2n-1} \left( \frac{x-1}{x+1} \right)^{2n-1} \quad (x > 0) \]

\[ \ln x = \frac{x-1}{x} + \frac{1}{2} \left( \frac{x-1}{x} \right)^2 + \frac{1}{3} \left( \frac{x-1}{x} \right)^3 + \ldots \]

\[ = \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{x-1}{x} \right)^n \quad (x > \frac{1}{2}) \]

\[ \ln x = (x-1) - \frac{1}{2} (x-1)^2 + \frac{1}{3} (x-1)^3 - \ldots \]

\[ = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} (x-1)^{n} \quad (0 < x \leq 2) \]

\[ \ln (1+x) = -\frac{1}{2} x^2 + \frac{1}{3} x^3 - \ldots \]

\[ = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} x^n \quad (|x| < 1) \]

\[ \log_e (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \infty \quad (-1 \leq x < 1) \]

\[ \log_e (1+x) - \log_e (1- x) = \]

\[ \log_e \left( \frac{1+x}{1-x} \right) = 2 \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \infty \right) (-1 < x < 1) \]

\[ \log_e \left( 1 + \frac{1}{n} \right) = \log_e \frac{n+1}{n} = 2 \left[ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots \infty \right] \]

\[ \log_{e} (1 + x) + \log_{e} (1 - x) = \log_{e} (1 - x^2) = -2 \left( \frac{x^2}{2} + \frac{x^4}{4} + \ldots \infty \right) (-1 < x < 1) \]

\[ \log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots \]
Important Results

(i) \( \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} \, dx = \frac{\pi}{4} \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} \, dx \)

(ii) \( \int_0^{\pi/2} \frac{\tan^n x}{1 + \tan^n x} \, dx = \frac{\pi}{4} \int_0^{\pi/2} \frac{1}{1 + \tan^n x} \, dx \)

(iii) \( \int_0^{\pi/2} \frac{\cot^n x}{1 + \cot^n x} \, dx = \frac{\pi}{4} \int_0^{\pi/2} \frac{\cot^n x}{1 + \cot^n x} \, dx \)

(iv) \( \int_0^{\pi/2} \frac{\tan^n x}{\tan^n x + \cot^n x} \, dx = \frac{\pi}{4} \int_0^{\pi/2} \frac{\cot^n x}{\tan^n x + \cot^n x} \, dx \) where, \( n \in \mathbb{R} \)

(v) \( \int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \csc^n x} \, dx = \frac{\pi}{4} \int_0^{\pi/2} \frac{\csc^n x}{\sec^n x + \csc^n x} \, dx \)

(vi) \( \int_0^{\pi/2} \frac{a^{\sin^n x}}{a^{\sin^n x} + a^{\cos^n x}} \, dx = \frac{\pi}{4} \int_0^{\pi/2} \frac{a^{\cos^n x}}{a^{\sin^n x} + a^{\cos^n x}} \, dx \)

(vii) \( \int_0^{\pi/2} \log \sin x \, dx = \frac{\pi}{2} \log 2 \)

(viii) \( \int_0^{\pi/2} \log \tan x \, dx = \frac{\pi}{2} \log 2 \)

(ix) \( \int_0^{\pi/2} \log \csc x \, dx = \frac{\pi}{2} \log 2 \)

(x) \( \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2} \)

(xi) \( \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \)

(xii) \( \int_0^{\infty} e^{-ax} x^n \, dx = \frac{n!}{a^n + 1} \)
\[
\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right) + C
\]
\[
\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + C
\]
\[
\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left(\frac{x-a}{x+a}\right) + C
\]
\[
\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right) + C
\]
\[
\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + C
\]
\[
\int \sqrt{a^2 + x^2} \, dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + C
\]
\[
\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cosh^{-1}\left(\frac{x}{a}\right) + C
\]
Good Luck to you for your Preparations, References, and Exams

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