My name is Subhashish Chattopadhyay. I have been teaching for IIT-JEE, Various International Exams (such as IMO [International Mathematics Olympiad], IPhO [International Physics Olympiad], IChO [International Chemistry Olympiad]), IGCSE (IB), CBSE, I.Sc, Indian State Board exams such as WB-Board, Karnataka PU-II etc since 1989. As I write this book in 2016, it is my 25th year of teaching. I was a Visiting Professor to BARC Mankhurd, Chembur, Mumbai, Homi Bhabha Centre for Science Education (HBCSE) Physics Olympics camp BARC Campus.
The selection for National Camp (for Official Science Olympiads - Physics, Chemistry, Biology, Astronomy) happens in the following steps ....

1 ) **NSEP** (National Standard Exam in Physics) and **NSEC** (National Standard Exam in Chemistry) held around 24th November. Approx 35,000 students appear for these exams every year. The exam fees is Rs 100 each. Since 1998 the IIT JEE toppers have been topping these exams and they get to know their rank / performance ahead of others.

2 ) **INPhO** (Indian National Physics Olympiad) and **INChO** (Indian National Chemistry Olympiad). Around 300 students in each subject are allowed to take these exams. Students coming from outside cities are paid fair from the Govt of India.

3 ) The Top 35 students of each subject are invited at HBCSE (Homi Bhabha Center for Science Education) Mankhurd, near Chembur, BARC, Mumbai. After a 2-3 weeks camp the top 5 are selected to represent India. The flight tickets and many other expenses are taken care by Govt of India.

Since last 50 years there has been no dearth of “Good Books“. Those who are interested in studies have been always doing well. This e-Book does not intend to replace any standard text book. These topics are very old and already standardized.
There are 3 kinds of Text Books

- **The thin Books** - Good students who want more details are not happy with these. Average students who need more examples are not happy with these. Most students who want to “Cram” quickly and pass somehow find the thin books “good” as they have to read less !!

- **The Thick Books** - Most students do not like these, as they want to read as less as possible. Average students are “busy” with many other things and have no time to read all these.

- **The Average sized Books** - Good students do not get all details in any one book. Most bad students do not want to read books of “this much thickness“ also !!

We know there can be no shoe that’s fits in all.

Printed books are not e-Books! Can’t be downloaded and kept in hard-disc for reading “later” .........

So if you read this book later, you will get all kinds of examples in a single place. This becomes a very good “Reference Material”. I sincerely wish that all find this “very useful”.

Students who do not practice lots of problems, do not do well. The rules of “doing well” had never changed .... Will never change !
After several students claimed that the Central Board of Secondary Education (CBSE) Class XII board Mathematics examination paper was 'tricky' and tough, the board has issued a clarification on remedial measures which are likely to be taken before evaluation.

The CBSE says that feedback received from various stakeholders like students, subject teachers and examiners will be put before the committee of subject experts.
On 21st May 2016 the CBSE standard 12 result was declared. I loved the headline

CBSE Class 12 Results out: No leniency in Maths paper, high paper standard to be maintained in future

The CBSE Class 12 Mathematics board exam on March 14 reduced many students to tears as they found the paper quite lengthy and tough and many couldn’t finish it on time. The results show an overall lowering of marks received in the Maths paper.

The CBSE (Central Board of Secondary Education) Class 12 Board exam results have been announced today, i.e on May 21, around 10:30 am ahead of time. Students may check their scores at the official website, [www.cbseresults.nic.in](http://www.cbseresults.nic.in). (Read: CBSE Class 12 Boards 2016: Results announced ahead of time! Check your score at cbseresults.nic.in)
In 2015 also the same complain was there by many students.

New Delhi: A senior Congress member on Thursday raised the issue of the tough mathematics question paper in the ongoing CBSE board examinations and asked the government to consider the issue "seriously".

So we see that by raising frivolous requests, even up to parliament, actually does not help. Many times requests from several quarters have been put to CBSE, or Parliament etc for easy Math Paper. These kinds of requests actually cannot be entertained, never will be.
In March 2016, students of Karnataka PU-II also complained the same, regarding standard 12 (PU-II Mathematics Exam). Even though the Math Paper was identical to previous year, most students had not even solved the 2015 Question Paper.

These complains are not new. In fact since last 40 years, (since my childhood), I always see this; every year the same setback, same complain!

In this e-Book I am trying to solve this problem. Those students who practice can learn.

No one can help those who are not studying, or practicing.
A very polite request:

I wish these e-Books are read only by Boys and Men. Girls and Women, better read something else; learn from somewhere else.
Preface

We all know that in the species “Homo Sapiens “, males are bigger than females. The reasons are explained in standard 10, or 11 ( high school ) Biology texts. This shapes or size, influences all of our culture. Before we recall / understand the reasons once again, let us see some random examples of the influence.

Random - 1

If there is a Road rage, then who all fight ? ( generally ? ). Imagine two cars driven by adult drivers. Each car has a woman of similar age as that of the Man. The cars “ touch “ or “ some issue happens”. Who all comes out and fights ? Who all are most probable to drive the cars ?

( Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win )

Random - 2

Heavy metal music artists are all Men. Metallica, Black Sabbath, Motley Crue, Megadeth, Motorhead, AC/DC, Deep Purple, Slayer, Guns & Roses, Led Zeppelin, Aerosmith ..... the list can be in thousands. All these are grown-up Boys, known as Men.

( Men strive for perfection. Men are eager to excel. Men work hard. Men want to win. )
Apart from Marie Curie, only one more woman got Nobel Prize in Physics. (Maria Goeppert Mayer - 1963). So, almost all are men.


The best Tabla Players are all Men.

History is all about, which Kings ruled. Kings, their men, and Soldiers went for wars. History is all about wars, fights, and killings by men.

Boys start fighting from school days. Girls do not fight like this.

( Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win. )
Random - 6

The highest award in Mathematics, the “Fields Medal” is around since decades. Till date only one woman could get that. (Maryam Mirzakhani - 2014). So, ... almost all are men.


Random - 7

Actor is a gender neutral word. Could the movie like “Top Gun” be made with Female actors? The best pilots, astronauts, Fighters are all Men.
In my childhood had seen a movie named “The Tower in Inferno”. In the movie when the tall tower is in fire, women were being saved first, as only one lift was working....

Many decades later another movie is made. A box office hit. “The Titanic”. In this also.... As the ship is sinking women are being saved. **Men are disposable.** Men may get their turn later...

Movies are not training programs. Movies do not teach people what to do, or not to do. Movies only reflect the prevalent culture. **Men are disposable,** is the culture in the society. Knowingly, unknowingly, the culture is depicted in Movies, Theaters, Stories, Poems, Rituals, etc. I or you can’t write a story, or make a movie in which after a minor car accident the Male passengers keep seating in the back seat, while the both the women drivers come out of the car and start fighting very bitterly on the road. There has been no story in this world, or no movie made, where after an accident or calamity, **Men are being helped for safety first, and women are told to wait.**
 Artists generally follow the prevalent culture of the Society. In paintings, sculptures, stories, poems, movies, cartoon, Caricatures, knowingly / unknowingly, “the prevalent Reality“ is depicted. The opposite will not go well with people. If deliberately “the opposite“ is shown then it may only become a special art, considered as a special mockery.

Men go to “girl / woman’s house” to marry / win, and bring her to his home. That is a sort of winning her. When a boy gets a “Girl-Friend“, generally he and his friends consider that as an achievement. The boy who “got / won“ a girl-friend feels proud. His male friends feel, jealous, competitive and envious. Millions of stories have been written on these themes. Lakhs of movies show this. Boys / Men go for “bike race“, or say “Car Race“, where the winner “gets“ the most beautiful girl of the college.

( Men want to excel. Men are eager to fight, eager to rule, eager for war. Men want to drive. Men want to win.)

Prithviraj Chauhan ‘went` to “pickup“ or “abduct“ or “win“ or “bring“ his love. There was a Hindi movie ( hit ) song ... “Pasand ho jaye, to ghar se utaha laye“. It is not other way round. Girls do not go to Boy’s house or man’s house to marry. Nor the girls go in a gang to “pick-up“ the boy / man and bring him to their home / place / den.
Rich people; often are very hard working. Successful business men, establish their business (empire), amass lot of wealth, with lot of difficulty. Lots of sacrifice, lots of hard work, gets into this. Rich people’s wives had no contribution in this wealth creation. Women are smart, and successful upto the extent to choose the right/rich man to marry. So generally what happens in case of Divorces? Search the net on “most costly divorces” and you will know. The women; (who had no contribution at all, in setting up the business / empire), often gets in Billions, or several Millions in divorce settlements.


See http://skmclasses.kinja.com/save-the-male-1761788732

It was Boys and Men, who brought the girls / women home. The Laws are biased, completely favoring women. The men are paying for their own mistakes.

See https://zookeepersblog.wordpress.com/biased-laws/

( Man brings the Woman home. When she leaves, takes away her share of big fortune! )

Random - 12

A standardized test of Intelligence will never be possible. It never happened before, nor ever will happen in future; where the IQ test results will be acceptable by all. In the net there are thousands of charts which show that the intelligence scores of girls / women are lesser. Debates of Trillion words, does not improve performance of Girls.
I am not wasting a single second debating or discussing with anyone, on this. I am simply accepting ALL the results. IQ is only one of the variables which is required for success in life. Thousands of books have been written on “Networking Skills”, EQ (Emotional Quotient), Drive, Dedication, Focus, “Tenacity towards the end goal “... etc. In each criteria, and in all together, women (in general) do far worse than men. Bangalore is known as “..... capital of India “. [Fill in the blanks]. The blanks are generally filled as “Software Capital”, “IT Capital”, “Startup Capital”, etc. I am member in several startup eco-systems / groups. I have attended hundreds of meetings, regarding “technology startups “, or “idea startups “. These meetings have very few women. Starting up new companies are all “Men’s Game” / “Men’s business “. Only in Divorce settlements women will take their goodies, due to Biased laws. There is no dedication, towards wealth creation, by women.

Random - 13

Many men, as fathers, very unfortunately treat their daughters as “Princess “. Every “non-performing “woman / wife was “princess daughter “ of some loving father. Pampering the girls, in name of “equal opportunity “, or “women empowerment “, have led to nothing.
There can be thousands of more such random examples, where “ Bigger Shape / size “ of males have influenced our culture, our Society. Let us recall the reasons, that we already learned in standard 10 - 11, Biology text Books. In humans, women have a long gestation period, and also spends many years ( almost a decade ) to grow, nourish, and stabilize the child. ( Million years of habit ) Due to survival instinct Males want to inseminate. Boys and Men fight for the “ facility ( of womb + care ) “ the girl / woman may provide. Bigger size for males, has a winning advantage. Whoever wins, gets the “ woman / facility “. The male who is of “ Bigger Size “, has an advantage to win…. Leading to Natural selection over millions of years. In general “ Bigger Males “; the “ fighting instinct “ in men; have led to wars, and solving tough problems ( Mathematics, Physics, Technology, startups of new businesses, Wealth creation, Unreasonable attempts to make things [ such as planes ], Hard work …. )

So let us see the IIT-JEE results of girls. Statistics of several years show that there are around 17, ( or less than 20 ) girls in top 1000 ranks, at all India level. Some people will yet not understand the performance, till it is said that … year after year we have around 980 boys in top 1000 ranks. Generally we see only 4 to 5 girls in top 500. In last 50 years not once any girl topped in IIT-JEE advanced. Forget about Single digit ranks, double digit ranks by girls have been extremely rare. It is all about “ good boys “, “ hard working “, “ focused “, “Bel-esprit “ boys.

In 2015, Only 2.6% of total candidates who qualified are girls ( upto around 12,000 rank ). while 20% of the Boys, amongst all candidates qualified. The Total number of students who appeared for the exam were around 1.4 million for IIT-JEE main. Subsequently 1.2 lakh ( around 120 thousands ) appeared for IIT-JEE advanced.

IIT-JEE results and analysis, of many years is given at https://zookeepersblog.wordpress.com/iit-jee-iseet-main-and-advanced-results/ In Bangalore it is rare to see a girl with rank better than 1000 in IIT-JEE advanced. We hardly see 6-7 boys with rank better than 1000. Hardly 2-3 boys get a rank better than 500.

See http://skmclasses.weebly.com/everybody-knows-so-you-should-also-know.html

Thousands of people are exposing the heinous crimes that Motherly Women are doing, or Female Teachers are committing. See https://www.facebook.com/WomenCriminals/
Some Random Examples must be known by all

It is extremely unfortunate that the "woman empowerment" has created. This is the kind of society and women we have now and many other sensible women hate such women. Be away from such women, be aware of reality.

‘Sex with my son is incredible - we’re in love and we want a baby’
Ben Ford, who ditched his wife when he met his mother Kim West after 30 years, claims what the couple are doing isn’t incest!

Woman sent to jail for raping her four grandchildren
A 39-year-old grandmother has been sentenced to four consecutive life terms after being found guilty of the rape of her own grandchildren. Emma Louise, 39, will spend the rest of her life behind bars.

CBSE Math Survival Guide - Hyperbola Coordinate Geometry by Prof. Subhashish Chattopadhyay
SKMClasses Bangalore Useful for IIT-JEE, I.Sc. PU-II, Boards, IGCSE IB AP-Mathematics and other exams
In several countries or rather in several regions of the world, family system has collapsed, due to bad nature and naughty acts of women. Particularly in Britain and America, almost 50% people are alone, lonely, separated, divorced or failed marriages. In 2013, 48% children were born out of wedlock. It was projected that by 2016, more than 51% children will be born, to unmarried mothers. In these developed countries “paternity fraud” by women, are close to 20%. You can see several articles in the net, and in wikipedia etc. This means 1 out of 5 children are calling a wrong man as dad. The lonely, alone “mothers” are frustrated. They see the children as burden. Love in the Society in general is lost, long time ago. The types of “Mothers “and “Women “we have now ............
By now if you have assumed that Indian women are not doing any crime then please become friends with MRA Gurj https://www.facebook.com/profile.php?id=100004138754180

He has dedicated his life to expose Indian Criminals
HURT FEMINISM BY DOING NOTHING

Don’t help women
Don’t fix things for women
Don’t support women’s issues
Don’t come to women’s defense
Don’t speak for women
Don’t value women’s feelings
Don’t portray women as victims
Don’t protect women

Without white knights feminism would end today

High Priority

Rich women
Women
Rich Men
Girls
Boys
Animals
Prisoners
Men
Poor Men

Low Priority

They can get away with murder.
They get all the rights with no responsibility and shelters for Homeless women.
They get bail outs and short prison sentences.
They get educational benefits but no violence against kids Act.
They have some support but don’t have any education that fits boys.
They have animal rights and PETA.
They get conjugal visits and a roof.
Paid slaves.
Nothing.

Who pays the most taxes?
This is why MGTOW exist.

Professor Subhashish Chattopadhyay
Spoon Feeding Series - Hyperbola

Three major types of conic sections:
**Horizontal Transverse Axis**

\[
\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1
\]

\[
y = -\frac{b}{a} \cdot x
\]

\[
y = \frac{b}{a} \cdot x
\]

**Vertical Transverse Axis**

\[
\frac{Y^2}{a^2} - \frac{X^2}{b^2} = 1
\]

\[
y = -\frac{a}{b} \cdot x
\]

\[
y = \frac{a}{b} \cdot x
\]
A hyperbola is an open curve with two branches, the intersection of a plane with both halves of a double cone. The plane does not have to be parallel to the axis of the cone, the hyperbola will be symmetrical in any case.
Here $a = b = 1$ giving the unit hyperbola in blue and its conjugate hyperbola in green, sharing the same red asymptotes.

Example: Write the equation in standard form of $4x^2 - 16y^2 = 64$. Find the foci and vertices of the hyperbola.

Divide both side of the equation by 64 to get the standard form:

$$\frac{4x^2}{64} - \frac{16y^2}{64} = \frac{64}{64}$$

Simplify...

$$\frac{x^2}{16} - \frac{y^2}{4} = 1$$

That means $a = 4$  $b = 2$

Use $c^2 = a^2 + b^2$ to find $c$.

$c^2 = 4^2 + 2^2$

$c^2 = 16 + 4 = 20$

$c = \sqrt{20} = 2\sqrt{5}$

Vertices: $(-4,0)$ and $(4,0)$

Foci: $(-2\sqrt{5},0)$ and $(2\sqrt{5},0)$
A hyperbola sort of looks like two parabolas that point at each other, and is the set of points whose distances from two fixed points (the foci) inside the ellipse is always the same, \(d_1 - d_2 = 2a\). The distance \(2a\) is called the focal radii distance, focal constant, or constant difference, and \(a\) is the distance between the center of the hyperbola to a vertex.

The equation of a “horizontal” hyperbola (as shown below) that is centered on the origin \((0, 0)\) is \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\). The length of the axis in which the hyperbola lies (called the transverse axis) is \(2a\), and this is along the \(x\)-axis for a horizontal hyperbola. Again, the distance from the center of the hyperbola to a vertex is \(a\), so the vertices are at \((\pm a, 0)\).

The length of the conjugate axis is \(2b\), and note that \(a\) does not have to be bigger than \(b\), like it does for an ellipse. (The distance from the center of the hyperbola to a co-vertex is \(b\)). Also note where the \(b\) is not on the hyperbola; it is on what we call the central rectangle (or fundamental rectangle) of the hyperbola (whose diagonals are asymptotes for the hyperbola). So the conjugate axis is along the \(y\)-axis for a horizontal hyperbola, and the co-vertices are at \((0, \pm b)\).

The asymptotes for a horizontal hyperbola centered at \((0, 0)\) are \(y = \pm \frac{b}{a}x\) (\(\pm \frac{b}{a}\) are the slopes, or the square root of what’s under the \(y\) over the square root of what’s under the \(x\)).

The asymptotes are the diagonals of the central rectangle of the hyperbola.

The focuses or foci always lie inside the curves on the major axis, and the distance from the center to a focus is \(c\). So the foci are at \((\pm c, 0)\) for a horizontal hyperbola (like an ellipse!), and it turns out that \(a^2 + b^2 = c^2\). (I like to remember that you always use the different sign for this equation: since ellipses have a plus sign in the equation \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\), they have a minus sign in \(a^2 + b^2 = c^2\); since hyperbolas have a minus sign in the equation \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\), they have a plus sign in \(a^2 + b^2 = c^2\).

Sometimes you will be asked to get the eccentricity of an hyperbola \(\frac{c}{a}\), which is a measure of how “straight” or “stretched” the hyperbola is.

Note also that, like for an ellipse, the focal width (focal chord, or focal rectum) of an ellipse is \(\frac{2b^2}{a}\), this the distance perpendicular to the major axis that goes through the focus.

Here is a horizontal hyperbola; we will also look at vertical and transformed hyperbolas below.
Here are the two different “directions” of hyperbolas and the generalized equations for each:

<table>
<thead>
<tr>
<th>Horizontal Hyperbola</th>
<th>Vertical Hyperbola</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^2 comes first)</td>
<td>(y^2 comes first)</td>
</tr>
<tr>
<td>At (0, 0):</td>
<td>At (0, 0):</td>
</tr>
<tr>
<td>[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 ]</td>
<td>[ \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 ]</td>
</tr>
<tr>
<td>General: [ \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 ]</td>
<td>General: [ \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 ]</td>
</tr>
<tr>
<td>Center: ((h, k))</td>
<td>Center: ((h, k))</td>
</tr>
<tr>
<td>Foci: ((h, k \pm c))</td>
<td>Foci: ((h, k \pm c))</td>
</tr>
<tr>
<td>Vertices: ((h \pm a, k))</td>
<td>Vertices: ((h, k \pm a))</td>
</tr>
<tr>
<td>Co-Vertices: ((h, k \pm b))</td>
<td>Co-Vertices: ((h \pm b, k))</td>
</tr>
<tr>
<td>Asymptotes: (y - k = \pm \frac{b}{a}(x - h))</td>
<td>Asymptotes: (y - k = \pm \frac{a}{b}(x - h))</td>
</tr>
</tbody>
</table>

Remember, for the conic to be a hyperbola, the coefficients of the \(x^2\) and \(y^2\) must have different signs.
Identify the center, vertices, foci, and equations of the asymptotes for the following hyperbolas; then graph: (a) \(9x^2 - 16y^2 - 144 = 0\) (b) \(\frac{(y+3)^2}{4} - \frac{(x-2)^2}{36} = 1\).

It’s typically easier to graph the hyperbola first, and then answer the questions.

<table>
<thead>
<tr>
<th>Graph</th>
<th>Math/Notes</th>
</tr>
</thead>
</table>
| (a) \(9x^2 - 16y^2 - 144 = 0\), which is \(\frac{x^2}{16} - \frac{y^2}{9} = 1\) | We first need to get our equation into the form of hyperbola by adding 144 to both sides, and then dividing all terms by 144; \(\frac{9x^2}{144} - \frac{16y^2}{144} = \frac{1}{16}\) or \(x^2 \over 16 \quad y^2 \over 9 = 1\). We will use the equation \(\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1\), since the \(x\) comes first (horizontal). This would make \(a^2 = 16\), so \(a = 4\). Since the hyperbola’s center is \((0, 0)\), the vertices are \((-4, 0)\) and \((4, 0)\).

\(b^2 = 9\), so \(b = 3\), so the co-vertices are \((0, -3)\) and \((0, 3)\). Now we can construct our central rectangle: we use \(a\) and \(b\) to create it.

Now let’s find the foci: \(c^2 = a^2 + b^2 = 9 + 16 = 25\). So \(c = 5\), and the foci are \((0, \pm 5)\).

The equation of the asymptotes (which go through the corners of the central rectangle) are \(y - k = \pm \frac{b}{a} (\text{rise}) (x - h)\), or \(y = \pm \frac{3}{4} x\). (Remember to use the square root of what’s under the \(y\) for the numerator of the slope, and the square of what’s under the \(x\) for the denominator.) |

| (b) \(\frac{(y+3)^2}{4} - \frac{(x-2)^2}{36} = 1\) | We will use equation: \(\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1\), since the \(y\) comes first (vertical).

We see that the center of the hyperbola is at \((2, -3)\), so we can first plot that point.

\(a^2 = 4\), so \(a = 2\). Since the center is \((2, -3)\), the vertices are \((2, -3 - 2)\) and \((2, -3 + 2)\), or \((2, -5)\) and \((2, -1)\).

\(b^2 = 36\), so \(b = 6\), so the co-vertices are \((2 - 6, -3)\) and \((2 + 6, -3)\) or \((-4, -3)\) and \((8, -3)\).

Now let’s find the foci: \(c^2 = a^2 + b^2 = 4 + 36 = 40\). So \(c = \sqrt{40}\) (or \(2\sqrt{10}\)), and the foci are \((2, -3 \pm \sqrt{40})\).

The equation of the asymptotes (which go through the corners of the central rectangle) are \(y - k = \pm \frac{a}{b} (\text{rise}) (x - h)\), or \(y + 3 = \frac{2}{6} (x - 2)\) or \(y + 3 = \frac{1}{3} (x - 2)\). (Remember to use the square root of what’s under the \(y\) for the numerator of the slope, and the square of what’s under the \(x\) for the denominator.) |
Identify the **center**, **vertices**, **foci**, and **equations of the asymptotes** for the following hyperbola; then **graph**: \(49y^2 - 25x^2 + 98y - 100x + 1174 = 0\).

<table>
<thead>
<tr>
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</tr>
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</table>
| \[
49y^2 - 25x^2 + 98y - 100x + 1174 = 0 \\
49y^2 + 98y - 25x^2 - 100x = -1174 \\
49\left(y^2 + \frac{2y}{9}\right) - 25\left(x^2 + \frac{4x}{9}\right) = -1174 + \frac{1}{9} \\
49\left(y^2 + \frac{1}{9}\right) - 25\left(x^2 + \frac{2}{9}\right) = -1174 + 49\left(\frac{1}{9}\right) - 25\left(\frac{2}{9}\right)^2 \\
49(y + 1)^2 - 25(x + 2)^2 = -1225 \\
\frac{(x+2)^2}{49} - \frac{(y+1)^2}{25} = 1 \\
\] | We need to first **complete the square** so we can get the equation in hyperbola form. Put all the \(x\)'s and \(y\)'s together with the constant term on the other side. Watch the negative signs; it turns out to be a horizontal hyperbola, with the coefficient of the \(x^2\) being positive (we ended up dividing all terms by \(-1225\)). We will use equation: \(\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1\), since \(x\) comes first (horizontal hyperbola). We see that the **center** of the hyperbola is at \((-2, -1)\), so we can first plot that point. \(a^2 = 49\), so \(a = 7\). Since the center is \((-2, -1)\), the **vertices** are \((-2 - 7, -1)\) and \((-2 + 7, -1)\), or \((-9, -1)\) and \((5, -1)\). \(b^2 = 25\), so \(b = 5\), so the **co-vertices** are \((-2, -1 - 5)\) and \((-2, -1 + 5)\) or \((-2, -6)\) and \((-2, 4)\). Now let's find the focc: \(c^2 = a^2 + b^2 = 49 + 25 = 74\). \(c = \sqrt{74}\), and the **foci** are \((-2 \pm \sqrt{74}, -1)\). The equation of the **asymptotes** (which go through the corners of the **central rectangle**) are \(y - k = \pm \frac{b}{a} \left(\frac{y-k}{a}\right)\), or \(y + 1 = \pm \frac{5}{7}(x + 2)\). |

Find the equation of the hyperbola where the difference of the **focal radii** is 6, and the endpoints of the **conjugate axis** are \((-2, 8)\) and \((-2, -2)\).

**Solution:**

We probably don't even need to graph this hyperbola, since we're basically given what \(a\) and \(b\) are. Remember that the **difference of the focal radii** is \(2a\), so \(a = 3\).

Since the endpoints of the conjugate axis are along a vertical line, we know that the hyperbola is **horizontal**, and the **co-vertices** are \((-2, 8)\) and \((-2, -2)\). From this information, we can get the center (midpoint between the co-vertices), which is \((-2, 3)\) and the length of the minor axis \(2b\), which is 10. So \(b = 5\). (Draw the points first if it’s difficult to see).

So the equation of the ellipse is \(\frac{(x+2)^2}{9} - \frac{(y-3)^2}{25} = 1\).
Find the equation of the hyperbola where one of the vertices is at \((-3, 2)\), and the asymptotes are \(y - 2 = \pm \frac{2}{3} (x - 3)\).

Solution:

Let’s try to graph this one, since it’s hard to tell what we know about it!

We can see from the equation of the asymptotes that the center of the hyperbola is \((3, 2)\).

Then we’ll graph this center and also graph the vertex that is given to see that the hyperbola is horizontal:

Now that we know the hyperbola is horizontal and we have the center and one vertex, we can see that \(a\) (difference between center and vertex) is 6. So far then we have:

\[
\frac{(x-3)^2}{6^2} - \frac{(y-2)^2}{b^2} = 1
\]

We also see from the asymptotes equation that their slope is \(\pm \frac{2}{3}\). We actually don’t even need to draw them, since we know in our case, since it’s a horizontal hyperbola, we’ll have \(y - 2 = \pm \frac{b}{6} (x - 3)\) (\(b\) is the square root of what’s under the \(y\); \(b\) is the square root of what’s under the \(x\) from the equation of the hyperbola above.

So now we can set up a proportion for the asymptote slopes:

\[
b = \frac{2}{3}\] by cross multiplying, we get \(b = 4\).

So the equation of the hyperbola is:

\[
\frac{(x-3)^2}{6^2} - \frac{(y-2)^2}{4^2} = 1,
\]
or

\[
\frac{(x-3)^2}{36} - \frac{(y-2)^2}{16} = 1.
\]
Rectangular Hyperbola

Eccentricity is $\sqrt{2}$ A hyperbola is said to be a rectangular hyperbola if its asymptotes are at right angles.

The angle between the asymptotes is given by $2 \tan^{-1} \left( \frac{b}{a} \right)$
Another form of Rectangular Hyperbola

Before we discuss examples and problems let us see the all the formulae

Image of a point

**The image of a point with respect to the line mirror.** The image of \(A(x_1, y_1)\) with respect to the line mirror \(ax + by + c = 0\) be \(B(h, k)\) given by,

\[
\begin{align*}
\frac{h-x_1}{a} &= \frac{k-y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2} \\
A(x_1, y_1) &\quad \quad ax + bx + c = 0 \\
B(h, k)
\end{align*}
\]
The image of a point with respect to x-axis: Let $P(x, y)$ be any point and $P'(x', y')$ its image after reflection in the x-axis, then

$x' = x$ and $y' = -y$, (\because \ O' \ is \ the \ mid \ point \ of \ PP'\)

The image of a point with respect to y-axis: $P(x, y)$ be any point and $P'(x', y')$ its image after reflection in the y-axis, then

$x' = -x$ and $y' = y$, (\because \ O' \ is \ the \ mid \ point \ of \ PP'\)
The image of a point with respect to the origin: Let \( P(x, y) \) be any point and \( P'(x', y') \) be its image after reflection through the origin, then
\[
x' = -x \quad \text{and} \quad y' = -y \quad (\because \ O \text{ is the mid-point of } PP')
\]

The image of a point with respect to the line \( y = x \): Let \( P(x, y) \) be any point and \( P'(x', y') \) be its image after reflection in the line \( y = x \), then,
\[
x' = y \quad \text{and} \quad y' = x \quad (\because \ O' \text{ is the mid-point of } PP')
\]
A Rhombus is made by distorting a square

All four sides are equal. So \( AB = BC = CD = DA \)
Area of a Triangle

The area of a triangle, the coordinates of whose vertices are \((x_1, y_1)\), \((x_2, y_2)\) and \((x_3, y_3)\) is given by

\[ \Delta = \frac{1}{2} \left[ x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right] \]

or

\[ \Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \]

Condition of collinearity of 3 points

Three points \(A(x_1, y_1), B(x_2, y_2)\) and \(C(x_3, y_3)\) are collinear if

i) Area of triangle \(ABC = 0\) i.e.,

\[ \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \]

or

ii) \(AB + BC = AC \) (or) \( AC + BC = AB \) (or) \( AC + AB = BC \)
In some cases a problem can be solved just by observation. Meaning the above determinant need not be evaluated.

The points \((a, b + c), (b, c + a)\) and \((c, a + b)\) are

(a) vertices of an equilateral triangle
(b) concyclic
(c) vertices of a right angled triangle
(d) none of these

**Ans. (d)**

**Solution** As the given points lie on the line \(x + y = a + b + c\), they are collinear.

Section formula Internal Division

Section formula External Division can have Two formulae. Depending on from which external side the division is being done
Here the external point Q is on the side of A

If m is the distance from A then m gets multiplied to coordinates of opposite point i.e.

\[ B(x_2, y_2) \]

The coordinates of the point Q which divides the line segment joining the points \( A(x_1, y_1) \) and \( B(x_2, y_2) \) externally in the ratio \( m:n \) are given by

\[
Q = \left( \frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n} \right)
\]

Note:

i) If P is the mid point of AB, then the coordinate of P is given by

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
\]

ii) The co-ordinate of any point on AB can be written as

\[
\left( \frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1} \right)
\]
Coordinates of the centroid, in-centre and ex-centres of a triangle

Let \( A(x_1, y_1) \), \( B(x_2, y_2) \) and \( C(x_3, y_3) \) be the three vertices of a triangle \( ABC \).

i) Centroid of a triangle

Centroid is the point of intersection of medians, whose coordinates are given by

\[
G = \left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)
\]

ii) In-centre of a triangle

In-centre is the point of intersection of internal angular bisectors, whose coordinates are given by

\[
I = \left( \frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c} \right)
\]
where $a$, $b$, $c$ are the lengths of the sides $BC$, $CA$, $AB$ respectively.

**iii) Ex-centres of a triangle**

The point of intersection $I_1$ of the external angular bisectors of $\angle B$ and $\angle C$ is one of the excentres of the triangle $ABC$ and is given by

$$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{a - b + c}, \frac{-ay_1 + by_2 + cy_3}{a - b + c}\right)$$

similarly the other excentres are given by

$$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a - b + c}, \frac{ay_1 - by_2 + cy_3}{a - b + c}\right)$$

$$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a + b - c}, \frac{ay_1 + by_2 - cy_3}{a + b - c}\right)$$

where $a$, $b$, $c$ are the lengths of the sides $BC$, $CA$, $AB$ respectively.

In some problems we find the Area pretty differently

The area of the triangle formed by the tangent to the curve $y = \frac{8}{4 + x^2}$ at $x = 2$ and the coordinate axes is

(a) $2$ sq. units  
(b) $\frac{7}{2}$ sq. units  
(c) $4$ sq. units  
(d) $8$ sq. units.
Solution

(c) From \( y = \frac{8}{4 + x^2} \),
when \( x = 2, y = \frac{8}{4 + 4} = 1 \)

Also, \( \frac{dy}{dx} = -\frac{8}{(4 + x^2)^2} (2x) \Rightarrow \left[ \frac{dy}{dx} \right]_{(2, 1)} = -\frac{1}{2} \)

\( \therefore \) equation of tangent is
\[ y - 1 = -\frac{1}{2} (x - 2) \text{ or } x + 2y - 4 = 0 \quad ...(1) \]

Its intercepts on axes are (by putting \( y = 0 \) and \( x = 0 \) respectively) \( a = 4, b = 2 \)

\( \therefore \) Area \( = \frac{1}{2} ab = \frac{1}{2} \times 4 \times 2 = 4 \text{ sq. units.} \)

- 

Perpendicular Lines

If there is a line whose slope is \( m \) (assuming this line NOT parallel to x-axis) then the slope of the line which is perpendicular to this will be \( -\frac{1}{m} \)

Meaning, product of the slopes of lines that are perpendicular is \( -1 \)

If one of the lines is parallel to x-axis its slope is 0 while the line perpendicular will have a slope of infinity (\( \infty \)). This line is parallel to y-axis. Product of 0 \( \times \infty \) is undefined. In this case we do not apply the \(-1\) as product rule.

-
Equation of the line passing through two points

\[ y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \]
The intercept form of a line

- Suppose a line $L$ makes $x$-intercept $a$ and $y$-intercept $b$ on the axes. Obviously $L$ meets $x$-axis at the point $(a, 0)$ and $y$-axis at the point $(0, b)$.

By two-point form of the equation of the line, we have

$$y - 0 = \frac{b - 0}{0 - a} (x - a)$$

Or

$$ay = -bx + ab$$

i.e.,

$$\frac{x}{a} + \frac{y}{b} = 1$$

Thus, equation of the line making intercepts $a$ and $b$ on $x$- and $y$-axis, respectively, is

$$\frac{x}{a} + \frac{y}{b} = 1$$

Question

Through the point $P(\alpha, \beta)$, where $\alpha \beta > 0$ the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is drawn so as to form with coordinate axes a triangle of area $S$. If $ab > 0$, then the least value of $S$ is

(a) $\alpha \beta$  
(b) $2\alpha \beta$  
(c) $4\alpha \beta$  
(d) none of these
Solution

(b). The equation of the given line is

\[ \frac{x}{a} + \frac{y}{b} = 1 \quad \ldots(1) \]

This line cuts the x-axis and y-axis at A \((a, 0)\) and B \((0, b)\) respectively.

Since area of \(\Delta OAB = S\) \hspace{1cm} (Given)

\[ \therefore \quad \frac{1}{2}ab = S \quad \text{or} \quad ab = 2S \quad (\because \ ab > 0) \quad \ldots(2) \]

Since the line (1) passes through the point \(P(\alpha, \beta)\),

\[ \therefore \quad \frac{\alpha}{a} + \frac{\beta}{b} = 1 \quad \text{or} \quad \frac{\alpha}{a} + \frac{a\beta}{2S} = 1 \quad \quad \text{[Using (2)]} \]

or \[ a^2 \beta - 2aS + 2\alpha S = 0. \]

Since \(a\) is real, \(\therefore \quad 4S^2 - 8\alpha \beta S \geq 0 \)

or \[ 4S^2 \geq 8\alpha \beta S \quad \text{or} \quad S \geq 2\alpha \beta \quad \left( \therefore \quad S = \frac{1}{2}ab > 0 \quad \text{as} \quad ab > 0 \right) \]

Hence the least value of \(S = 2\alpha \beta.\)
i) The equation of a line parallel to a given line \( ax+by+c=0 \) is \( ax+by+\lambda=0 \), where \( \lambda \) is constant.

ii) The equation of a line perpendicular to a given line \( ax+by+c=0 \) is \( bx-ay+\lambda=0 \), where \( \lambda \) is constant.

iii) The slope of the line \( ax+by+c=0 \) is given by
\[
\frac{m}{b} = -\frac{a}{b}
\]

iv) For intercept on x-axis, put \( y=0 \). For intercept on y-axis, put \( x=0 \).

v) Angle \( \theta \) between the lines \( a_1x+b_1y+c_1=0 \), \( a_2x+b_2y+c_2=0 \) is given by
\[
\tan \theta = \left| \frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2} \right|
\]

vi) The lines \( a_1x+b_1y+c_1=0 \), \( a_2x+b_2y+c_2=0 \) are

a) Coincident if \( \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \)

b) Parallel if \( \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \)

c) Intersecting if \( \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \)

d) Perpendicular if \( a_1a_2 + b_1b_2 = 0 \)
Distance of a point from a line

The length of the perpendicular from a point \((x_1, y_1)\) to a line \(ax + by + c = 0\) is given by

\[ PN = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \]

Note:
The length of the perpendicular from the origin to the line \(ax + by + c = 0\) is

\[ \frac{|c|}{\sqrt{a^2 + b^2}} \]
The distance between the parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is given by

$$\frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

The two points $(x_1, y_1)$ and $(x_2, y_2)$ are on the same (or opposite) sides of the straight line $ax + by + c = 0$ according to the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same (or opposite) signs.

The three lines $a_1 x + b_1 y + c_1 = 0$, $a_2 x + b_2 y + c_2 = 0$ and $a_3 x + b_3 y + c_3 = 0$ are concurrent (intersect at a point) if and only if

$$\begin{vmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3 \\
\end{vmatrix} = 0$$
The equations of the straight lines which pass through a given point \((x_1, y_1)\) and make a given angle \(\alpha\) with the given straight line \(y = mx + c\) are

\[
y - y_1 = \frac{m \pm \tan \alpha}{1 \mp m \tan \alpha} (x - x_1)
\]

The angle between the lines \(x \cos \alpha_1 + y \sin \alpha_1 = P_1\) and \(x \cos \alpha_2 + y \sin \alpha_2 = P_2\) is \(\alpha_1 - \alpha_2\).
Equation of Internal and External bisectors of 2 Lines

The equation of the bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Bisector of the angle containing the origin

If $c_1, c_2$ are positive, then the equation of the bisector of the angle containing the origin is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$
Bisector of Acute and Obtuse angle between lines

i) If \( c_1, c_2 \) are positive and if \( a_1a_2 + b_1b_2 > 0 \), then
\[
\frac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2 x + b_2 y + c_2}{\sqrt{a_2^2 + b_2^2}}
\]

is the obtuse angle bisector and
\[
\frac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = \mp \frac{a_2 x + b_2 y + c_2}{\sqrt{a_2^2 + b_2^2}}
\]

is the acute angle bisector.

ii) If \( c_1, c_2 \) are positive and if \( a_1a_2 + b_1b_2 < 0 \), then
\[
\frac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2 x + b_2 y + c_2}{\sqrt{a_2^2 + b_2^2}}
\]

is the acute angle bisector and
\[
\frac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} = \mp \frac{a_2 x + b_2 y + c_2}{\sqrt{a_2^2 + b_2^2}}
\]

is the obtuse angle bisector.

If \( c_1, c_2 \) are positive and \( a_1a_2 + b_1b_2 > 0 \), then the origin lies in the obtuse angle and the ‘+’ sign gives the bisector of the obtuse angle.

If \( a_1a_2 + b_1b_2 < 0 \), then the origin lies in the acute angle and ‘+’ sign gives the bisector of acute angle.
Coordinates of Centroid, Orthocenter, Circumcenter of a Triangle

Centroid: The point of intersection of the medians of a triangle is called its centroid. It divides the median in the ratio 2:1.

If \((x_1, y_1), (x_2, y_2),\) and \((x_3, y_3)\) are the vertices of a triangle, then the coordinates of its centroid are

\[
\left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)
\]
Orthocentre

The point of intersection of the altitudes of a triangle is called its orthocentre.

To determine the orthocentre, first we find equations of line passing through vertices and perpendicular to the opposite sides. Solving two of these three equations we get the co-ordinates of orthocentre.

If angles $A$, $B$ and $C$ and vertices $A\ (x_1, y_1)$, $B\ (x_2, y_2)$ and $C\ (x_3, y_3)$ of a $\Delta ABC$ are given, then orthocentre of $\Delta ABC$ is given by

$$
\left( \frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C} \right)
$$
If any two lines out of three lines, i.e., $AB$, $BC$ and $CA$ are perpendicular, then orthocentre is the point of intersection of two perpendicular lines. The orthocentre of the triangle with vertices $(0, 0)$, $(x_1, y_1)$ and $(x_2, y_2)$ is
\[
\left\{ \begin{array}{l}
(y_1 - y_2) \left[ \frac{x_1 x_2 - y_1 y_2}{x_2 y_1 - x_1 y_2} \right] \\
(x_1 - x_2) \left[ \frac{x_1 x_2 + y_1 y_2}{x_1 y_2 - x_2 y_1} \right]
\end{array} \right.
\]

Question on Orthocenter

The orthocentre of the triangle formed by the lines $xy = 0$ and $2x + 3y - 5 = 0$ is
(a) $(2, 3)$  
(b) $(3, 2)$  
(c) $(0, 0)$  
(d) $(5, -5)$

**Ans.** (c)

**Solution** The given triangle is right angled at $(0, 0)$ which is therefore the orthocentre of the triangle.

**Circumcentre**

The point of intersection of the perpendicular bisectors of the sides of a triangle is called its circum-centre. It is equidistant from the vertices of a triangle.
Question

If the circumcentre of a triangle lies at the origin and the centroid is the middle point of the line joining the points \((a^2 + 1, a^2 + 1)\) and \((2a, -2a)\); then the orthocentre lies on the line

(a) \(y = (a^2 + 1)x\)  
(b) \(y = 2ax\)  
(c) \(x + y = 0\)  
(d) \((a - 1)^2 x - (a + 1)^2 y = 0\)

Ans. (d)

Solution  We know from geometry that the circumcentre, centroid and orthocentre of a triangle lie on a line. So the orthocentre of the triangle lies on the line joining the circumcentre \((0, 0)\) and the centroid \(\left(\frac{(a+1)^2}{2} , \frac{(a-1)^2}{2}\right)\)

i.e. \(\frac{(a+1)^2}{2} y = \frac{(a-1)^2}{2} x\)

or \((a - 1)^2 x - (a + 1)^2 y = 0\).
The equations to the sides of a triangle are \( x - 3y = 0 \), \( 4x + 3y = 5 \) and \( 3x + y = 0 \). The line \( 3x - 4y = 0 \) passes through the
(a) incentre
(b) centroid
(c) circumcentre
(d) orthocentre of the triangle

**Ans.** (d)

**Solution** Two sides \( x - 3y = 0 \) and \( 3x + y = 0 \) of the triangle being perpendicular to each other, the triangle is right angled at the origin, the point of intersection of these sides. So that origin is the orthocentre of the triangle and the line \( 3x - 4y = 0 \) passes through this orthocentre.

**Ex-Centres of a Triangle** A circle touches one side outside the triangle and the other two extended sides then circle is known as excircle.

Let \( ABC \) be a triangle then there are three excircles, with three exc centres \( I_1, I_2, I_3 \) opposite to vertices \( A, B \) and \( C \) respectively.

If the vertices of triangle are \( A (x_1, y_1) \), \( B (x_2, y_2) \) and \( C (x_3, y_3) \) then

\[
I_1 = \left( \frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c} \right)
\]

\[
I_2 = \left( \frac{ax_1 - bx_2 + cx_3}{a - b + c}, \frac{ay_1 - by_2 + cy_3}{a - b + c} \right)
\]

\[
I_3 = \left( \frac{ax_1 + bx_2 - cx_3}{a + b - c}, \frac{ay_1 + by_2 - cy_3}{a + b - c} \right)
\]
Family of lines through the intersection of two given lines

The equation of family of lines passing through the intersection of the lines

\[ L_1 = a_1x + b_1y + c_1 = 0 \quad \text{and} \quad L_2 = a_2x + b_2y + c_2 = 0 \]

is

\[ (a_1x + b_1y + c_1) + \lambda (a_2x + b_2y + c_2) = 0, \text{where } \lambda \]

is a parameter i.e., \( L_1 + \lambda L_2 = 0 \).

Formulae specific to Pair of Straight Lines

Homogeneous equation of second degree in \( x \) and \( y \)

A general homogeneous equation of degree 2 always represent two straight lines, real or imaginary, through the origin. Conversely, the equal of a pair of lines through origin is a second degree homogeneous equation in \( x \) and \( y \).

The equation of the form \( ax^2 + 2hxy + by^2 = 0 \) is called a homogeneous equation of degree 2 in \( x \) and \( y \), where \( a, b, h \) are constants.

let \( ax^2 + 2hxy + by^2 = 0 \) \( \ldots (1) \)

\[ b\left(\frac{y^2}{x^2}\right) + 2h\left(\frac{y}{x}\right) + a = 0 \]

The general equation \( ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \) represents a pair of Straight lines only if

\[ abc + 2fga - af^2 - bg^2 - ch^2 = 0 \quad \text{i.e., iff } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \]

For easy remembering note that the first row of the Determinant is coeffs of \( x \) terms

\[ (a)x^2 + 2(h)xy \ldots + 2 (g) x \ldots . \]
Similarly the second row is made of coeffs of y terms. i.e.

\[2 (h) xy + (b)y^2 + 2 (f) y \ldots\]

The last row of the determinant is the last 3 constants of last 3 terms. i.e. g, f, and c.

---

**Equation of the lines joining the origin to the points of intersection of a line and a conic.**

Let

\[L = lx + my + n = 0\]

and

\[S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\]

be the equations of a line and a conic, respectively. Writing the equation of the line as \(lx + my\) \(-n\) = 1 and making \(S = 0\) homogeneous with its help, we get

\[S = ax^2 + 2hxy + by^2 + 2(gx + fy) \left(\frac{lx + my}{-n}\right) + c \left(\frac{lx + my}{-n}\right)^2 = 0\]

which being a homogeneous equation of second degree, represents a pair of straight lines through the origin and passing through the points common to \(S = 0\) and \(L = 0\).

---

**Equation of the pair of lines through the origin perpendicular to the pair of lines** \(ax^2 + 2hxy + by^2 = 0\) **is** \(bx^2 - 2hxy + ay^2 = 0\).

---

**Question**

If the slope of one of the lines represented by \(ax^2 - 6xy + y^2 = 0\) is square of the other, then

- (a) \(a = 1\)
- (b) \(a = 2\)
- (c) \(a = 4\)
- (d) \(a = 8\)

**Ans.** (d)

**Solution** Let the lines represented by the given equation be \(y = mx\) and \(y = m^2x\), then

\[m + m^2 = 6 \text{ and } m^3 = a\]

\[\Rightarrow m = 2 \text{ or } -3\]

and so

\[a = 8 \text{ or } -27\]
So the equation of the required lines is
\[(x + 3y) (3x + y) = 0 \Rightarrow 3x^2 + 10xy + 3y^2 = 0.\]

Question on Locus

If \(P (1, 0), Q (-1, 0)\) and \(R (2, 0)\) are three given points.
The point \(S\) satisfies the relation \(SQ^2 + SR^2 = 2SP^2\). The locus of \(S\) meets \(PQ\) at the point
(a) \((0, 0)\)  \hspace{1cm} (b) \((2/3, 0)\)
(c) \((-3/2, 0)\) \hspace{1cm} (d) \((0, -2/3)\)

\textbf{Ans.} (c)

\textbf{Solution} \hspace{0.5cm} \text{Let } S \text{ be the point } (x, y)\n\text{then } \( (x + 1)^2 + y^2 + (x - 2)^2 + y^2 = 2[(x - 1)^2 + y^2 ]\)
\[\Rightarrow 2x + 3 = 0, \text{ the locus of } S \text{ and equation of } PQ \text{ is } y = 0.\]
So the required points is \((-3/2, 0)\).
Formulae related to circles

The line $y = mx + c$ intersects the circle $x^2 + y^2 = a^2$ at two distinct points if the length of the perpendicular from the centre is less than the radius of the circle.

$$\left| \frac{c}{\sqrt{1 + m^2}} \right| < a$$

Equation of a line tangent to the circle at a point $(x_2, y_2)$ on the circle

Example: Find the tangent line equation at point $(1,2)$ to the circle:

$$x^2 + y^2 + 2x + 3y - 13 = 0$$

$$m = \frac{y_2 - a}{b - y_2} = \frac{4}{7} = -0.57$$

$$y = \frac{4}{7}x + \frac{18}{7}$$

and finally $7y = -4x + 18$

Circle form:

$$(x - a)^2 + (y - b)^2 = r^2$$

Tangent line slope:

$$m = \frac{x_2 - a}{b - y_2}$$

tangent line equation:

$$y = \frac{x_2 - a}{b - y_2} (x - x_2) + y_2$$

provided that $b \neq y_2$

If $b = y_2$ then the line equation becomes:

$$x = x_2$$

Circle form:

$$x^2 + y^2 + Ax + By + C = 0$$

$x_2, y_2$ should be a point on the circle therefore:

$$x_2^2 + y_2^2 + Ax_2 + By_2 + C = 0$$

Tangent line slope:

$$m = -\frac{2x_2 + A}{2y_2 + B}$$

tangent line equation:

$$y = -\frac{2x_2 + A}{2y_2 + B} (x - x_2) + y_2$$
The line does not intersect the circle \( x^2 + y^2 = a^2 \) if the length of the perpendicular from the centre is greater than the radius of the circle.

\[
\left| \frac{c}{\sqrt{1+m^2}} \right| > a
\]

iii) The length of the intercept cut off from a line \( y = mx + c \) by a circle \( x^2 + y^2 = a^2 \) is
2MB = 2 \sqrt{\frac{a^2 (1 + m^2) - c^2}{1 + m^2}}

Inscribed angles
(1) All inscribed angles intercepted by the same arc or chord and lies on the same side of the cord are equal.
(2) Sum of opposite angles drawn from the same cord are equal to 180°, \( \alpha + \theta = 180° \)
(3) If the cord coincides with the diameter of the circle then the inscribed angle is a right angle, \( \alpha = \theta = 90° \).

Central and inscribed angles
If central angle \( \theta \) and inscribed angle \( \alpha \) intercepts the same cord or arc then:
\[ \theta = 2\alpha \quad \text{or} \quad \alpha = \frac{\theta}{2} \]
**Intersecting secants theorem**

If a line from a point $P$ intersects the circle at two different locations then:

$$\overline{PA} \cdot \overline{PB} = \overline{PC} \cdot \overline{PD}$$

We can also write: $(a + b) a = (c + d) c$

If line $PD = c$ is tangent then $c^2 = (a + b) a$.

---

**Tangent and secant line theorem**

The value of an angle formed by a secant or tangent line drawn from a point $P$ outside the circle equal the half of the difference of the intercepted arcs or central angles.

$$\alpha = \frac{1}{2}(L_1 - L_2) = \frac{1}{2} (\theta_1 - \theta_2)$$

$\theta_1 \theta_2$ are the central angles of the shown arcs.

---

**Cyclic quadrilateral**

The sum of the opposite angles in an inscribed (cyclic) quadrilateral are equal to $180^\circ$.

$$\alpha + \gamma = 180^\circ$$

$$\beta + \delta = 180^\circ$$

---

**Ptolemy's theorem**

The product of the diagonals of a cyclic quadrilateral equals the sum of the products of the opposite sides.

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$

---

**Intersecting chord theorem**

The product of the two segments created by intersecting of two chords are equal.

$$\overline{AO} \cdot \overline{OC} = \overline{BO} \cdot \overline{OD}$$
Arc Segment and Sector

<table>
<thead>
<tr>
<th>Circle arc</th>
<th>(θ in radians)</th>
<th>(θ in degree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L(r, \theta) = r\theta )</td>
<td>( L = \text{Arc length} )</td>
<td>( L = \frac{\pi \theta}{180} )</td>
</tr>
<tr>
<td>( L(r, c) = 2r \sin^{-1} \left( \frac{c}{2r} \right) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( L(r, t) = 2r \cos^{-1} \left( \frac{r-t}{r} \right) )</td>
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</tr>
</tbody>
</table>

Example: Find the tangency point of the circles: \((x - 4)^2 + (y + 1)^2 = 9\) and \((x + 1)^2 + (y + 1)^2 = 4\)

Check if circles are tangent:
\((-4 - 1)^2 + (1 - 1)^2 = (3 + 2)^2\)
5 = 5 (result with the plus sign, so the circles are tangent to each other)

Tangency point coordinate:
\[
x = \frac{(a - c)(r_0^2 - r_1^2)}{2[(c - a)^2 + (d - b)^2]} \cdot \frac{a + c}{2}
\]
\[
y = \frac{(b - d)(r_0^2 - r_1^2)}{2[(c - a)^2 + (d - b)^2]} \cdot \frac{b + d}{2}
\]

Condition for tangency of two circles:
\[(a - c)^2 + (b - d)^2 = (r_0 \pm r_1)^2\]
(+ sign is for external tangency and − for internal tangency)

If we have the two radii \(r_0\) and \(r_1\) and the distance between the centers \(d\), then conditions for tangency are:

Outer circles tangency:
\[r_0 + r_1 = d\]

Inner circles tangency:
\[|r_0 - r_1| = d\]
<table>
<thead>
<tr>
<th>Circle sector</th>
<th>(\theta in radians)</th>
<th>(\theta in degree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A(r, \theta) = \frac{\theta}{2} r^2)</td>
<td>(A = \frac{\theta}{360} \pi r^2)</td>
<td></td>
</tr>
<tr>
<td>(A(r, L) = \frac{L}{2} r)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(A(r, c) = r^2 \sin^{-1} \left( \frac{c}{2r} \right))</td>
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</tr>
<tr>
<td>(P(r, \theta) = r(\theta + 2))</td>
<td>(P = r \left( \frac{\theta}{180} \pi + 2 \right))</td>
<td></td>
</tr>
<tr>
<td>(P(r, c) = 2r \left( \sin^{-1} \left( \frac{c}{2r} \right) + 1 \right))</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Circle segment</th>
<th>(\theta in radians)</th>
<th>(\theta in degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A(r, \theta) = \frac{r^2}{2} (\theta - \sin \theta))</td>
<td>(A = \frac{r^2}{2} \left( \frac{\pi}{180} \theta - \sin \theta \right))</td>
<td></td>
</tr>
<tr>
<td>(A(r, c) = r^2 \sin^{-1} \left( \frac{c}{2r} \right) - \frac{c}{4} \sqrt{4r^2 - c^2})</td>
<td></td>
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<tr>
<td>(A(r, h) = r^2 \cos^{-1} \left( \frac{h}{r} \right) - h \sqrt{r^2 - h^2})</td>
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<tr>
<td>(A(r, t) = r^2 \cos^{-1} \left( \frac{t}{r} \right) - (r - t) \sqrt{r^2 - 2rt})</td>
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</tr>
<tr>
<td>(P(r, \theta) = r(\theta + 2 \sin \frac{\theta}{2}))</td>
<td>(P = r \left( \frac{\pi}{180} \theta + 2 \sin \frac{\theta}{2} \right))</td>
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<tr>
<td>(P(r, c) = r^2 \sin^{-1} \left( \frac{c}{2r} \right) - \frac{c}{2} \sqrt{4r^2 - c^2})</td>
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</table>

Segment sector relations

Area (A), Circumference (P)

\[A_{\text{segment}} = A_{\text{sector}} - A_{\text{triangle}}\]

<table>
<thead>
<tr>
<th>(f(r, \theta))</th>
<th>(f(r, h))</th>
<th>(f(r, t))</th>
<th>(f(r, c))</th>
<th>(f(h, c))</th>
<th>(f(h, t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c = 2r \sin \frac{\theta}{2})</td>
<td>(2r \sqrt{r^2 - h^2})</td>
<td>(2r \sqrt{r^2 - t^2})</td>
<td>(-)</td>
<td>(-)</td>
<td>(2r \sqrt{t^2 + 2ht})</td>
</tr>
<tr>
<td>(t = r \left(1 - \cos \frac{\theta}{2}\right))</td>
<td>(r - h)</td>
<td>(-)</td>
<td>(r - \frac{1}{2} \sqrt{4r^2 - c^2})</td>
<td>(\frac{1}{2} \sqrt{4h^2 + c^2} - h)</td>
<td>(-)</td>
</tr>
<tr>
<td>(h = r \cos \frac{\theta}{2})</td>
<td>(-)</td>
<td>(r - t)</td>
<td>(\frac{1}{2} \sqrt{4r^2 - c^2})</td>
<td>(-)</td>
<td>(-)</td>
</tr>
<tr>
<td>(r = -)</td>
<td>(-)</td>
<td>(-)</td>
<td>(-)</td>
<td>(\frac{1}{2} \sqrt{4h^2 + c^2})</td>
<td>(h + t)</td>
</tr>
<tr>
<td>(\theta = -)</td>
<td>(2 \cos^{-1} \left( \frac{h}{r} \right))</td>
<td>(2 \cos^{-1} \left( \frac{t - 1}{r} \right))</td>
<td>(2 \sin^{-1} \left( \frac{c}{2r} \right))</td>
<td>(2 \tan^{-1} \left( \frac{c}{2h} \right))</td>
<td>(2 \cos^{-1} \left( \frac{h}{h + t} \right))</td>
</tr>
<tr>
<td>(A_1 = \frac{r^2}{2} \sin \theta)</td>
<td>(h \sqrt{r^2 - h^2})</td>
<td>(r - t) \sqrt{r^2 - t^2})</td>
<td>(\frac{c}{4} \sqrt{4r^2 - c^2})</td>
<td>(\frac{1}{2} ch)</td>
<td>(h \sqrt{r^2 + 2th})</td>
</tr>
<tr>
<td>(P_1 = r \left(2 \sin \frac{\theta}{2} + 2\right))</td>
<td>(2r + \sqrt{r^2 - h^2})</td>
<td>(2r + \sqrt{2r^2 - t^2})</td>
<td>(2r + c)</td>
<td>(c + \sqrt{4h^2 + c^2})</td>
<td>(2 \left( h + t + \sqrt{t^2 + 2th} \right))</td>
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<tr>
<td>$c = \frac{2h \tan \frac{\theta}{2}}{\cos \frac{\theta}{2}}$</td>
<td>$-\frac{c}{2 \tan \frac{\theta}{2}}$</td>
<td>$-\frac{c^2 - 4t^2}{8t}$</td>
<td>$-\frac{t \cos \frac{\theta}{2}}{1 - \cos \frac{\theta}{2}}$</td>
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</tr>
<tr>
<td>$t = \left(\frac{1}{\cos \frac{\theta}{2}}\right) \left(\frac{1 - \cos \frac{\theta}{2}}{\sin \frac{\theta}{2}}\right)$</td>
<td>$-\frac{c}{2 \sin \frac{\theta}{2}}$</td>
<td>$-\frac{4t^2 + c^2}{8t}$</td>
<td>$-\frac{t}{1 - \cos \frac{\theta}{2}}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$h = -\frac{h}{\cos \frac{\theta}{2}}$</td>
<td>$\left(\frac{1}{\sin \frac{\theta}{2}}\right)$</td>
<td>$\left(\frac{1}{\sin \frac{\theta}{2}} + 1\right)$</td>
<td>$\frac{2t \left(1 + \sin \frac{\theta}{2}\right)}{1 - \cos \frac{\theta}{2}}$</td>
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<td></td>
</tr>
<tr>
<td>$r = \frac{c}{2 \sin \frac{\theta}{2}}$</td>
<td>$-\frac{c}{2 \sin \frac{\theta}{2}}$</td>
<td>$\frac{4t^2 + 2}{8t}$</td>
<td>$-\frac{t}{1 - \cos \frac{\theta}{2}}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta = -\frac{c}{2 \sin \frac{\theta}{2}}$</td>
<td>$-\frac{c}{2 \sin \frac{\theta}{2}}$</td>
<td>$-\frac{c}{2 \sin \frac{\theta}{2}}$</td>
<td>$-\frac{c}{2 \sin \frac{\theta}{2}}$</td>
<td></td>
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</tr>
<tr>
<td>$A_t = \frac{h^2 \tan \frac{\theta}{2}}{2}$</td>
<td>$\frac{c^2}{2 \tan \frac{\theta}{2}}$</td>
<td>$\frac{c}{16t} \left(c^2 - 4t^2\right)$</td>
<td>$\frac{c^2 - 4t^2}{2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P_t = \frac{2h \left(1 + \sin \frac{\theta}{2}\right)}{\sin \frac{\theta}{2} \cos \frac{\theta}{2}}$</td>
<td>$\frac{c}{2 \sin \frac{\theta}{2}}$</td>
<td>$\frac{c + 4t^2 + c^2}{4t}$</td>
<td>$\frac{2t \left(1 + \sin \frac{\theta}{2}\right)}{1 - \cos \frac{\theta}{2}}$</td>
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</tbody>
</table>

**Question on Tangent**

The point on the curve $y = 6x - x^2$ where the tangent is parallel to x-axis is

(a) (0, 0)  (b) (2, 8)  (c) (6, 0)  (d) (3, 9).

**Solution**

(d) $\frac{dy}{dx} = 6 - 2x$

$\therefore \frac{dy}{dx} = 0 \Rightarrow x = 3.$

$\therefore y = 18 - 9 = 9 \therefore$ Point is (3, 9).
For the curve \( x = t^2 - 1, \) \( y = t^2 - t, \) the tangent line is perpendicular to \( x \)-axis, where
\[
(a) \ t = 0 \quad \quad \quad \quad \quad \quad (b) \ t \to \infty \\
(c) \ t = \frac{1}{\sqrt{3}} \quad \quad \quad \quad \quad \quad (d) \ t = -\frac{1}{\sqrt{3}}.
\]

**Solution**

\[
(a) \ \frac{dx}{dt} = 2t, \\
\text{Tangent is perpendicular to \( x \)-axis if } \frac{dx}{dt} = 0 \Rightarrow t = 0.
\]

**Question**

The point on the curve \( y^2 = x, \) the tangent at which makes an angle of 45° with \( x \)-axis will be given by
\[
(a) \left( \frac{1}{2}, \frac{1}{4} \right) \quad \quad \quad (b) \left( \frac{1}{2}, \frac{1}{2} \right) \\\n(c) \ (2, 4) \quad \quad \quad \quad \quad (d) \left( \frac{1}{4}, \frac{1}{2} \right).
\]

**Solution**

\[
(d) \ y^2 = x \Rightarrow 2y \frac{dy}{dx} = 1 \\
\Rightarrow \frac{dy}{dx} = \frac{1}{2y} = \tan 45^\circ = 1 \ (\text{given}) \\
\Rightarrow y = \frac{1}{2} \quad \quad \Rightarrow x = \frac{1}{4} \\
\therefore \ \text{Point is } \left( \frac{1}{4}, \frac{1}{2} \right).
\]
Question

If tangent to the curve $x = at^2, y = 2at$ is perpendicular to $x$-axis then its point of contact is

(a) $(a, a)$  
(b) $(0, a)$  
(c) $(a, 0)$  
(d) $(0, 0)$.

Solution

$\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a \Rightarrow \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

$\Rightarrow \frac{1}{t} = \infty \Rightarrow t = 0$ \Rightarrow Point is $(0, 0)$.

Equation of the circle when the end points of a diameter are given

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter of circle and let $P$ be any point on circle.
Now, since the angle subtended at the point P in the semicircle APB is a right angle,

\[ m_1 m_2 = -1 \quad (m_1 = \text{slope of AP, } m_2 = \text{slope of BP}) \]

\[ \frac{y - y_1}{x - x_1} \frac{y - y_2}{x - x_2} = -1 \]

\[ \text{i.e., } (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0 \]

**Condition for two intersecting circles to be orthogonal**

**Definition**

Two intersecting circles are said to cut each other orthogonally when the tangents at the point of intersection of the two circles are at right angles.

Let the circles

\[ S_1 = x^2 + y^2 + 2g_1x + 2f_1y + C_1 = 0 \text{ and } \]

\[ S_2 = x^2 + y^2 + 2g_2x + 2f_2y + C_2 = 0 \]
intersect orthogonally, then $\angle C_1P C_2 = 90^\circ$

ie., $\Delta C_1 P C_2$ is right angled

$\therefore C_1 C_2^2 = C_1 P^2 + C_2 P^2$

$(g_1 - g_2)^2 + (f_1 - f_2)^2 = (g_1^2 + f_1^2 - c_1) + (g_2^2 + f_2^2 - c_2)$

$\Rightarrow 2g_1g_2 + 2f_1f_2 = c_1 + c_2$ is the required condition that $S_1$ and $S_2$ intersect orthogonally.
Some important results

i) The equation of chord joining two points \(\theta_1\) and \(\theta_2\) on the circle \(x^2 + y^2 + 2gx + 2fy + c = 0\) is

\[
(x + g) \cos \frac{\theta_1 + \theta_2}{2} + (y + f) \sin \frac{\theta_1 + \theta_2}{2} = r
\]

\[
\cos \frac{\theta_1 - \theta_2}{2}, \text{ where } r \text{ is the radius of the circle.}
\]

ii) The equation of the tangent at \(P(\theta)\) on the circle \(x^2 + y^2 + 2gx + 2fy + c = 0\) is \((x + g)\)

\[
\cos \theta + (y + f) \sin \theta = \sqrt{g^2 + f^2 - c}
\]

iii) The locus of the point of intersection of two tangents drawn to the circle \(x^2 + y^2 = a^2\)
which makes an constant angle \(\alpha\) to each other is \(x^2 + y^2 - 2a^2 = 4a^2(x^2 + y^2 - a^2)\cot^2\alpha\).

Question

The equation of tangent to the circle \(x^2 + y^2 + 6x + 4y - 12 = 0\) at \((6,2)\) is

a) \(4x - 9y - 6 = 0\)  b) \(9x + 4y + 12 = 0\)

b) \(3x - 9y = 0\)  d) \(2x - 3y = 6\)

Ans (b)

Note:

The equation of tangent at \((x_1, y_1)\) is

\[
x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0
\]

thus the equation of tangent at \((6,2)\) is

\[
6x + 2y + 3(x + 6) + 2(y + 2) - 12 = 0
\]

i.e., \(9x + 4y + 12 = 0\).
Question on Angle of intersection

The angle of intersection of the curves $y = x^2$ and $6y = 7 - x^3$ at $(1, 1)$ is

(a) $\frac{\pi}{4}$  
(b) $\frac{\pi}{3}$  
(c) $\frac{\pi}{2}$  
(d) None of these.

Solution

(c) $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow m_1 = 2$

$6y = 7 - x^3 \Rightarrow \frac{dy}{dx} = -\frac{1}{2} \Rightarrow m_2 = -\frac{1}{2}$

$\therefore \quad m_1m_2 = -1$ at $(1, 1)$

$\Rightarrow \quad \theta = \frac{\pi}{2}$.

Question

If $a, x_1, x_2$ are in G.P. with common ratio $r$, and $b, y_1, y_2$ are in G.P. with common ratio $s$ where $s - r = 2$, then the area of the triangle with vertices $(a, b), (x_1, y_1)$ and $(x_2, y_2)$ is

(a) $|ab (r^2 - 1)|$  
(b) $ab (r^2 - s^2)$  
(c) $ab (s^2 - 1)$  
(d) $abrs$

Ans. (a)

Solution  Area of the triangle

\[
\frac{1}{2} \begin{vmatrix} a & b & 1 \\ ar & bs & 1 \\ ar^2 & bs^2 & 1 \end{vmatrix} = \frac{1}{2} |ab (r-1)(s-1)(s-r)|
\]

\[
= |ab (r - 1) (r + 1)| = |ab (r^2 - 1)|
\]
ELLIPSE

An ellipse is the locus of a point which moves in a plane so that the ratio of its distance from a fixed point (called focus) and a fixed line (called directrix) is a constant which is less than one. This ratio is called eccentricity and is denoted by $e$. For an ellipse, $e < 1$.

Let $S$ be the focus, $QN$ be the directrix and $P$ be any point on the ellipse. Then, by definition, $\frac{PS}{PN} = e$ or $PS = e \cdot PN$, $e < 1$, where $PN$ is the length of the perpendicular from $P$ on the directrix $QN$.

An alternate definition An ellipse is the locus of a point that moves in such a way that the sum of its distances from two fixed points (called foci) is constant.
EQUATION OF AN ELLIPSE IN STANDARD FORM

The standard form of the equation of an ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b),$$

where $a$ and $b$ are constants.

SOME TERMS AND PROPERTIES RELATED TO AN ELLIPSE

A sketch of the locus of a moving point satisfying the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b),$$

has been shown in the figure given above.
1. Symmetry
   (a) On replacing $y$ by $-y$, the above equation remains unchanged. So, the curve is symmetrical about $x$-axis
   (b) On replacing $x$ by $-x$, the above equation remains unchanged. So, the curve is symmetrical about $y$-axis

2. Foci
   If $S$ and $S'$ are the two foci of the ellipse and their coordinates are $(ae, 0)$ and $(-ae, 0)$ respectively, then distance between foci is given by
   $$SS' = 2ae.$$

3. Directrices
   If $ZM$ and $Z'M'$ are the two directrices of the ellipse and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively, then the distance between directrices is given by
   $$ZZ' = \frac{2a}{e}.$$
4. **Axes** The lines $AA'$ and $BB'$ are called the **major axis** and **minor axis** respectively of the ellipse.

The length of major axis $= AA' = 2a$

The length of minor axis $= BB' = 2b$

5. **Centre** The point of intersection $C$ of the axes of the ellipse is called the centre of the ellipse. All chords, passing through $C$ are bisected at $C$.

6. **Vertices** The end points $A$ and $A'$ of the major axis are known as the **vertices** of the ellipse.

$$A = (a, 0) \text{ and } A' = (-a, 0)$$

**Remember:** The vertex divides the join of focus and the point of intersection of directrix with axis internally and externally the ratio $e : 1$.

7. **Focal chord** A chord of the ellipse passing through its focus is called a **focal chord**.

8. **Ordinate and double ordinate** Let $P$ be a point on the ellipse. From $P$, draw $PN \perp AA'$ (major axis of the ellipse) and produce $PN$ to meet the ellipse at $P'$. Then $PN$ is called an **ordinate** and $PNP'$ is called the **double ordinate** of the point $P$.

9. **Latus rectum** If $LL'$ and $NN'$ are the latus rectum of the ellipse, then these lines are $\perp$ to the major axis $AA'$, passing through the foci $S$ and $S'$ respectively.

$$L = \left( ae, \frac{b^2}{a} \right), \quad L' = \left( ae, -\frac{b^2}{a} \right),$$

$$N = \left( -ae, \frac{b^2}{a} \right), \quad N' = \left( -ae, -\frac{b^2}{a} \right)$$

Length of latus rectum $= LL' = \frac{2b^2}{a} = NN'$.

10. By definition, $SP = ePM = e \left( \frac{a}{e} - x \right) = a - ex$

and $S'P = e \left( \frac{a}{e} + x \right) = a + ex$.

This implies that distances of any point $P(x, y)$ lying on the ellipse from foci are: $(a - ex)$ and $(a + ex)$. In other words
11. Eccentricity of the ellipse
Since, \( SP = ePM \), therefore,

\[
SP^2 = e^2PM^2
\]

or

\[
(x - ae)^2 + (y - 0)^2 = e^2\left(\frac{a}{e} - x\right)^2
\]

\[
x^2 + a^2e^2 - 2aex + y^2 = a^2 - 2aex + e^2x^2
\]

\[
x^2(1 - e^2) + y^2 = a^2(1 - e^2)
\]

\[
\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.
\]

On comparing with \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), we get

\[
b^2 = a^2(1 - e^2) \quad \text{or} \quad e = \sqrt{1 - \frac{b^2}{a^2}}.
\]

12. Auxiliary circle
The circle drawn on major axis \( AA' \) as diameter is known as the Auxiliary circle.

Let the equation of the ellipse be \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). Then the equation of its auxiliary circle is:

\[
x^2 + y^2 = a^2.
\]
Let $Q$ be a point on auxiliary circle so that $QM$, perpendicular to major axis meets the ellipse at $P$. The points $P$ and $Q$ are called as corresponding points on the ellipse and auxiliary circle respectively.

The angle $\theta$ is known as eccentric angle of the point $P$ on the ellipse.

It may be noted that the $CQ$ and not $CP$ is inclined at $\theta$ with $x$-axis.

13. **Parametric equation of the ellipse** The coordinates $x = a \cos \theta$ and $y = b \sin \theta$ satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

for all real values of $\theta$. Thus, $x = a \cos \theta, y = b \sin \theta$ are the parametric equations of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where the parameter $0 \leq \theta < 2\pi$.

Hence the coordinates of any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

may be taken as $(a \cos \theta, b \sin \theta)$. This point is also called the point '$\theta$'.

The angle $\theta$ is called the eccentric angle of the point $(a \cos \theta, b \sin \theta)$ on the ellipse.

14. **Equation of Chord** The equation of the chord joining the points $P = (a \cos \theta_1, b \sin \theta_1)$ and $Q = (a \cos \theta_2, b \sin \theta_2)$ is

$$\frac{x}{a} \cos \left(\frac{\theta_1 + \theta_2}{2}\right) + \frac{y}{b} \sin \left(\frac{\theta_1 + \theta_2}{2}\right) = \cos \left(\frac{\theta_1 - \theta_2}{2}\right).$$
Remember: If the centre of the ellipse lies at \((h, k)\) and the axes are parallel to the coordinate axes, then the equation of the ellipse is

\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1
\]

POSITION OF A POINT WITH RESPECT TO AN ELLIPSE

The point \(P(x_1, y_1)\) lies outside, on or inside the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) according as \(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > 0, = 0, \text{ or } < 0\).

Intersection of line and an Ellipse

The line \(y = mx + c\) intersects the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) in two distinct points if \(a^2m^2 + b^2 > c^2\), in one point if \(c^2 = a^2m^2 + b^2\) and does not intersect if \(a^2m^2 + b^2 < c^2\).

CONDITION FOR TANGENCY AND POINTS OF CONTACT

The condition for the line \(y = mx + c\) to be a tangent to the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) is that \(c^2 = a^2m^2 + b^2\) and the coordinates of the points of contact are

\[
\left( \pm \frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \pm \frac{b}{\sqrt{a^2m^2 + b^2}} \right)
\]
Two standard forms of the ellipse

- **Horizontal Form of an Ellipse**
  \[
  \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)
  \]

- **Vertical Form of an Ellipse**
  \[
  \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (a, b)
  \]

### Centre
- **Horizontal Form**: \((0, 0)\)
- **Vertical Form**: \((0, 0)\)

### Equation of major axis
- **Horizontal Form**: \(y = 0\)
- **Vertical Form**: \(x = 0\)

### Equation of minor axis
- **Horizontal Form**: \(x = 0\)
- **Vertical Form**: \(y = 0\)

### Length of major axis
- **Horizontal Form**: \(2a\)
- **Vertical Form**: \(2a\)

### Length of minor axis
- **Horizontal Form**: \(2b\)
- **Vertical Form**: \(2b\)

### Foci
- **Horizontal Form**: \((\pm ae, 0)\)
- **Vertical Form**: \((0, \pm ae)\)

### Vertices
- **Horizontal Form**: \((\pm a, 0)\)
- **Vertical Form**: \((0, \pm a)\)

### Equation of directrices
- **Horizontal Form**: \(x = \pm \frac{a}{e}\)
- **Vertical Form**: \(y = \pm \frac{a}{e}\)

### Eccentricity
- **Horizontal Form**: \(e = \sqrt{\frac{a^2 - b^2}{a^2}}\)
- **Vertical Form**: \(e = \sqrt{\frac{a^2 - b^2}{a^2}}\)

### Length of latus rectum
- **Horizontal Form**: \(\frac{2b^2}{a}\)
- **Vertical Form**: \(\frac{2b^2}{a}\)

### Ends of latus-recta
- **Horizontal Form**: \((\pm ae, \pm \frac{b^2}{a})\)
- **Vertical Form**: \((\pm \frac{b^2}{a}, \pm ae)\)

### Parametric coordinates
- **Horizontal Form**: \((a \cos \theta, b \sin \theta)\)
- **Vertical Form**: \((a \cos \theta, b \sin \theta)\)

### Focal radii
- **Horizontal Form**: \(SP = a - ex\) and \(S'P = a + ex\)
- **Vertical Form**: \(SP = a - ey\) and \(S'P = a + ey\)

### Sum of focal radii \(SP + S'P\)
- **Horizontal Form**: \(2a\)
- **Vertical Form**: \(2a\)

### Distance between foci
- **Horizontal Form**: \(2ae\)
- **Vertical Form**: \(2ae\)

### Distance between directrices
- **Horizontal Form**: \(\frac{2a}{e}\)
- **Vertical Form**: \(\frac{2a}{e}\)

### Tangents at the vertices
- **Horizontal Form**: \(x = \pm a\)
- **Vertical Form**: \(y = \pm a\)
Formulae related to ellipse

The equation of tangent to the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) at \( P(x_1y_1) \) is \( \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \)

The equation of normal to the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) at \( P(x_1y_1) \) is \( \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 \)

Note:
Four normals can be drawn from any point to the ellipse.
Condition for $y = mx + c$ to be a tangent to the ellipse and points of tangency

The equation of tangent to the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) at \( P(x_1, y_1) \) is
\[
\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \ldots(1)
\]

Given \(mx + y = c\) \ldots(2)

(1) and (2) represent the same line

\[
\frac{x_1}{a^2} = \frac{y_1}{b^2} = \frac{1}{c}
\]

\[
-m \cdot \frac{1}{c} = 1 \quad \Rightarrow x_1 = \frac{-a^2m}{c}, \quad y_1 = \frac{b^2}{c}
\]

Since \(P(x_1, y_1)\) lies on the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\)

we get,

\[
\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1
\]

\[
\Rightarrow \frac{a^4m^2}{c^2a^2} + \frac{b^4}{c^2b^2} = 1
\]

**CHORD WITH A GIVEN MID POINT**

The equation of the chord of the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) with

\(P(x_1, y_1)\) as its middle point is given by

\[T = S_1\]

where \(T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\) and \(S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\).
So Review the formulae

The following are some standard results for an ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) and a hyperbola \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \):

1. The parametric equations of an ellipse (hyperbola) or the coordinates of any point on the ellipse (hyperbola) are \( x = a \cos \theta, y = b \sin \theta \) \( (x = a \sec \theta, \ y = b \tan \theta) \). The point is denoted “\( \theta \) “.

2. An equation of the tangent at the above point “\( \theta \) “ is

\[
\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1 \quad \left( \frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1 \right)
\]

3. An equation of the normal at the same point “\( \theta \) “ is

\[
\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2 \quad \left( \frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2 \right)
\]
4. An equation of the tangent at the point \( P(x', y') \) on the ellipse is

\[
\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1
\]

For the hyperbola \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \), results corresponding to (4) – (6) and (8) are obtained by replacing \( b^2 \) by \((-b^2)\).

5. The condition that the line \( y = mx + c \) touches the ellipse is \( c^2 = a^2 m^2 + b^2 \), so that the equation of any tangent to the ellipse (not parallel to the \( y \)-axis) can be written as \( y = mx \pm \sqrt{a^2 m^2 + b^2} \).

6. **Director circle** of an ellipse is the locus of the point of intersection of tangents to the ellipse which intersect at right angles and its equation is \( x^2 + y^2 = a^2 + b^2 \).

7. **Auxiliary circle** of an ellipse is the circle on major axis of the ellipse as diameter and its equation is \( x^2 + y^2 = a^2 \).

If \( P \) is a point on the ellipse and \( Q \) is a point on the auxiliary circle such that \( Q \) lies on the ordinate produced of the point \( P \), then \( \angle ACQ \) (where \( CA \) is the semimajor axis of the ellipse) is called the **eccentric angle** of the point \( P \) on the ellipse and the coordinates of \( P \) are \((a \cos \phi, b \sin \phi)\) where \( \phi = \angle ACQ \).
8. A diameter of an ellipse is the locus of the mid points of a system of parallel chords of the ellipse and its equation is

\[ y = -\frac{b^2}{a^2} m x, \]

where \( m \) is the slope of the parallel chords of the ellipse which are bisected by it. This is a line through the centre of the ellipse. Two diameters of an ellipse are said to be *conjugate* when each bisects the chords parallel to the others. Thus two diameters \( y = m x \) and \( y = m'x \) of the ellipse are conjugate if

\[ mm' = -\frac{b^2}{a^2}. \]

9. A hyperbola whose asymptotes are perpendicular to each other is called a *rectangular hyperbola* and its equation is \( x^2 - y^2 = a^2 \). By taking the asymptotes of the rectangular hyperbola as the coordinate axes, its equation can be written as \( x y = c^2 \) (where \( c^2 = a^2/2 \)) and the parametric equation of this rectangular hyperbola is \( x = c t, y = c/t, t \) being the parameter.

An asymptote to a curve is a line which touches the curve at infinity. Thus equation of the asymptotic of the hyperbola

\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

is

\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \]
Question

The number of values of c such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4} + y^2 = 1$ is

(a) 0  (b) 1  (c) 2  (d) infinite

Ans. (c)

Solution We know that $y = mx + c$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$

Here $m = a^2 = 4, b^2 = 1$ so $c^2 = 4 \times 4^2 + 1 \Rightarrow c = \pm \sqrt{65}$

Question

The focii of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then the value of $b^2$ is

(a) 5  (b) 7  (c) 9  (d) 1

Ans. (b)

Solution $16 - b^2 = \frac{144}{25} + \frac{81}{25} \Rightarrow b^2 = 7$.

Question

The normal to the curve at $P(x, y)$ meets the x-axis at $G$. If the distance of $G$ from the origin is twice the abscissa of $P$, then the curve is

(a) ellipse  (b) parabola
(c) circle  (d) hyperbola or ellipse
Ans. (d)

Solution  Equation of the normal at \((x, y)\) is \(Y - y = -\frac{dx}{dy} (X - x)\) which meets the \(x\)-axis at \(G\left(x + y \frac{dy}{dx}\right)\), then \(x + y \frac{dy}{dx} = \pm 2x\)

\[\Rightarrow \quad x + y \frac{dy}{dx} = 2x \quad \Rightarrow \quad y \ dy = x \ dx \quad \Rightarrow \quad x^2 - y^2 = c\]

or \[\Rightarrow \quad y \ dy = -3x \ dx \quad \Rightarrow \quad 3x^2 + y^2 = c\]

Thus the curve is either hyperbola or ellipse.

Formulae related to Hyperbola

Parametric equations of the hyperbola

A point \((x, y)\) on the hyperbola \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\) can be represented as \(x = a \ \sec \ \theta, \ y = b \ \tan \ \theta\) in a single parameter \(\theta\). These equations are called parametric equations of the hyperbola \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\). The point \((a \ \sec \ \theta, b \ \tan \ \theta)\) is simply denoted by \(\theta\).
Some important results

i) The equation of the chord joining the points
\((a \sec \alpha, b \tan \alpha)\) and \((a \sec \beta, b \tan \beta)\) is
\[
\frac{x}{a} \cos \frac{\alpha - \beta}{2} - \frac{y}{b} \sin \frac{\alpha + \beta}{2} = \cos \frac{\alpha + \beta}{2}.
\]

ii) The equation of the tangent at \(P(\theta)\) on the
hyperbola \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\) is
\[
\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1.
\]

iii) The equation of the normal at \(P(\theta)\) on the
hyperbola \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\) is
\[
\frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2.
\]
iv) The condition that the line $lx + my + n = 0$
may be a normal to the hyperbola
\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}
\]

v) If $P$ is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci $S$ and $S'$, then $S'P - SP = 2a$.

vi) The locus of point of intersection of perpendicular tangents to an hyperbola
\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is a circle } x^2 + y^2 = a^2 - b^2 \text{ called director circle of the hyperbola.}
\]

vii) The locus of the feet of perpendiculars drawn the foci to any tangent to the hyperbola
\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is a circle } x^2 + y^2 = a^2, \text{ called auxiliary circle of the hyperbola.}
\]
Parabola

\[ y^2 = 4ax \text{ is a standard form of the equation of a parabola.} \]

Four standard forms of a parabola are:

- \[ x = -a \]
- \[ x = a \]
- \[ y = -a \]
- \[ y = a \]

Parabolas have two shapes: concave up or concave down.

- **Concave-up parabola:** "a" is positive in \( ax^2 + bx + c \)
- **Concave-down parabola:** "a" is negative in \( ax^2 + bx + c \)

**Examples of concave up equations**
- \[ y = 3x^2 + 2x + 1 \]
- \[ y = x^2 - 3x \]
- \[ y = 6x^2 + 2 \]
- \[ y = 5x^2 \]

**Examples of concave down equations**
- \[ y = -3x^2 + 2x + 1 \]
- \[ y = -x^2 - 3x \]
- \[ y = -6x^2 + 2 \]
- \[ y = -5x^2 \]
The following terms are used in context of the parabola \( y^2 = 4ax \).

1. The point \( O(0, 0) \) is the vertex of the parabola, and the tangent to the parabola at the vertex is \( x = 0 \).
2. The line joining the vertex \( O \) and the focus \( S(a, 0) \) is the axis of the parabola and its equation is therefore \( y = 0 \).
3. Any chord of the parabola perpendicular to its axis is called a double ordinate.
4. Any chord of the parabola passing through its focus is called a focal chord.
5. The focal chord of the parabola perpendicular to its axis is called its latus rectum; the length of this latus rectum is therefore \( 4a \).
6. The points on a parabola, the normals at which are concurrent, are called co-normal points of the parabola. If \((x_1, y_1), (x_2, y_2)\) and \((x_3, y_3)\) are conormal points of the parabola \( y^2 = 4ax \), then \( y_1 + y_2 + y_3 = 0 \).
7. A line which bisects a system of parallel chords of a parabola is called a diameter of the parabola.
<table>
<thead>
<tr>
<th>Vertical Parabola</th>
<th>Horizontal Parabola</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Positive Coefficient</strong></td>
<td><strong>Positive Coefficient</strong></td>
</tr>
<tr>
<td>At $(0,0)$: $y = ax^2$</td>
<td>At $(0,0)$: $x = ay^2$</td>
</tr>
<tr>
<td><strong>General</strong>: $y = a(x-h)^2 + k$ or $y - k = a(x-h)^2$</td>
<td><strong>General</strong>: $x = a(y-k)^2 + h$ or $x - h = a(y-k)^2$</td>
</tr>
<tr>
<td>$y = \frac{1}{4a}(x-h)^2 + k$ or $y - k = \frac{1}{4a}(x-h)^2$</td>
<td>$x = \frac{1}{4a}(y-k)^2 + h$ or $x - h = \frac{1}{4a}(y-k)^2$</td>
</tr>
<tr>
<td>or $4a(y-k) = (x-h)^2$</td>
<td>or $4a(x-h) = (y-k)^2$</td>
</tr>
<tr>
<td><strong>Vertex</strong>: $(h, k)$</td>
<td><strong>Vertex</strong>: $(h, k)$</td>
</tr>
<tr>
<td><strong>Axis of Symmetry</strong>: $x = h$</td>
<td><strong>Axis of Symmetry</strong>: $y = k$</td>
</tr>
</tbody>
</table>

The equations for the vertical and horizontal parabolas are shown with diagrams illustrating the parabolas and their axes of symmetry.
The following are some standard results for the parabola $y^2 = 4ax$:

1. The parametric equations of the parabola or the coordinates of any point on it are $x = at^2$, $y = 2at$.
2. The tangent to the parabola at $(x', y')$ is $yy' = 2a(x + x')$ and that at $(at^2, 2at)$ is $ty = x + at^2$.
3. The condition that the line $y = mx + c$ is a tangent to the parabola is $c = alm$ and the equation of any tangent to it (not parallel to the $y$-axis) is therefore $y = m(x + (alm))$.
4. The chord of contact (defined as in circles) of $(x', y')$ w.r.t. the parabola is $yy' = 2a(x + x')$.
5. The polar (defined as in circle) of $(x', y')$ w.r.t. the parabola is $yy' = 2a(x + x')$.
6. The chord with mid-Point $(x', y')$ of the parabola is $T = S'$, where $T = yy' - 2a(x + x')$ and $S' = y'^2 - 4ax'$.
7. The equation of the pair of tangents from $(x', y')$ to the parabola is $T^2 = SS'$. Where $S = y^2 - 4ax$.
8. The normal at $(at^2, 2at)$ to the parabola is $y = -tx + 2at + at^2$. If $m$ is the slope of this normal, then its equation is $y = mx - 2am - am^2$, which is the normal to the parabola at $(am^2, -2am)$.
9. A diameter of the parabola is the locus of the middle points of a system of parallel chords of the parabola and the equation of a diameter is $y = 2alm$ where $m$ is the slope of the parallel chords which are bisected by it.
10. The equation of a chord joining $(at^1, 2at_1)$ and $(at^2, 2at_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$.
11. If the chord joining the points having parameters $t_1$ and $t_2$ passes through the focus, then $t_1t_2 = -1$.
12. If the coordinates of one end of a focal chord are $(at^2, 2at)$, then the coordinates of the other end are $(at^2, -2at)$.
13. For the end of the latus rectum, the values of the parameters $t$ are $\pm 1$.
14. The tangents at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ intersect at $(at_1t_2, a(t_1 + t_2))$.
15. The tangents at the extremities of any focal chord intersect at right angles on the directrix.
16. The locus of the point of intersection of perpendicular tangents to the parabola is its directrix.
17. The area of the triangle formed by any three points on the parabola is twice the area of the triangle formed by the tangents at these points.
18. The circle described on any focal chord of a parabola as diameter touches the directrix.

**OPTICAL PROPERTY OF PARABOLA**

(a) A ray parallel to the axis of the parabola after reflection from its internal surface passes through the focus.

(b) If a point is at a minimum distance from a parabola, then this point must lie on a normal to the parabola through this point.
The point of intersection of tangents drawn at two different points of contact \( P(at_1^2, 2at_1) \) and \( Q(at_2^2, 2at_2) \) on the parabola \( y^2 = 4ax \) is

\[ R = (at_1t_2, a(t_1 + t_2)). \]
\[
\left( \frac{2at_1 + 2at_2}{2} \right) = \frac{a(t_1 + t_2)}{2}
\]

is the y-coordinate of the point of intersection of tangents at \( P \) and \( Q \) on the parabola.

The orthocentre of the triangle formed by three tangents to the parabola lies on the directrix.

The locus of the point of intersection of tangents to the parabola \( y^2 = 4ax \) which meet at an angle \( \alpha \) is

\[
(x + \alpha)^2 \tan^2 \alpha = y^2 - 4ax
\]

The tangents to the parabola \( y^2 = 4ax \) at \( P(\,at_1^2, 2at_1)\) and \( Q(\,at_2^2, 2at_2)\) intersect at \( R \). Then the area of triangle

\[
PQR = \frac{1}{2}a^2(t_1 - t_2)^3.
\]

If the straight line \( lx + my + n = 0 \) touches the parabola \( y^2 = 4ax \), then \( ln = am^2 \).
If the line \( \frac{x}{l} + \frac{y}{m} = 1 \) touches the parabola \( y^2 = 4a(x + b) \) then \( m^2(l + b) + 4a^2 = 0 \).

If the two parabolas \( y^2 = 4x \) and \( x^2 = 4y \) intersect at point \( P \), whose abscissa is not zero, then the tangent to each curve at \( P \), make complementary angle with the \( x \)-axis.

If the line \( x \cos \alpha + y \sin \alpha = p \) touches the parabola \( y^2 = 4ax \), then \( p \cos \alpha + a \sin^2 \alpha = 0 \) and the point of contact is \( (a \tan^2 \alpha, -2a \tan \alpha) \).

Tangents at the extremities of any focal chord of a parabola meet at right angle on the directrix.

Area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

If the tangents at the points \( P \) and \( Q \) on a parabola meet in \( T \), then \( ST \) is the geometric mean between \( SP \) and \( SQ \), i.e., \( ST^2 = SP \cdot SQ \).

**POSITION OF A POINT WITH RESPECT TO A PARABOLA**

The point \((x_1, y_1)\) lies outside, on or inside the parabola \( y^2 = 4ax \) according as \( y_1^2 - 4ax_1 >, = \) or \(< 0 \), respectively.

**NUMBER OF TANGENTS DRAWN FROM A POINT TO A PARABOLA**

Two tangents can be drawn from a point to a parabola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the parabola.
EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point \( P(x_1, y_1) \) to the parabola \( y^2 = 4ax \) is \( SS_1 = T^2 \),
where \( S = y^2 - 4ax, S_1 = y_1^2 - 4ax_1 \)
and \( T = yy_1 - 2a(x + x_1) \)

\[
\begin{align*}
Q & \\
R & \\
P & (x_1, y_1)
\end{align*}
\]

EQUATIONS OF NORMAL IN DIFFERENT FORMS

1. **Point Form**  The equation of the normal to the parabola \( y^2 = 4ax \) at a point \((x_1, y_1)\) is

\[
y - y_1 = -\frac{y_1}{2a} (x - x_1).
\]

2. **Parametric Form**  The equation of the normal to the parabola \( y^2 = 4ax \) at the point \((at^2, 2at)\) is

\[
y + tx = 2at + at^3.
\]

3. **Slope Form**  The equation of normal to the parabola \( y^2 = 4ax \) in terms of slope ‘m’ is

\[
y = mx - 2am - am^3.
\]

*Note: The coordinates of the point of contact are \((am^2, -2am)\).*

**Condition for Normality**  The line \( y = mx + c \) is a normal to the parabola

\[
y^2 = 4ax \text{ if } c = -2am - am^3.
\]
POINT OF INTERSECTION OF NORMALS

The point of intersection of normals drawn at two different points of contact \( P(at_1^2, 2at_1) \) and \( Q(at_2^2, 2at_2) \) on the parabola \( y^2 = 4ax \) is

\[
R = [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)].
\]

If the normal at the point \( P(at_1^2, 2at_1) \) meets the parabola \( y^2 = 4ax \) again at \( Q(at_2^2, 2at_2) \), then

\[
t_2 = -\frac{t_1}{t_2} - \frac{2}{t_1},
\]

Note that \( PQ \) is normal to the parabola at \( P \) and not at \( Q \).
If the normals at the points \((at_1^2, 2at_1)\) and \((at_2^2, 2at_2)\) meet on the parabola \(y^2 = 4ax\), then \(t_1t_2 = 2\).

**CO-NORMAL POINTS**

Any three points on a parabola normals at which pass through a common point are called co-normal points.

If three normals are drawn through a point \((h, k)\), then their slopes are the roots of the cubic:

\[
k = mh - 2am - am^3
\]

(i) The sum of the slopes of the normals at co-normal points is zero, i.e. \(m_1 + m_2 + m_3 = 0\).

(ii) The sum of the ordinates of the co-normal points is zero (i.e. \(-2am_1 - 2am_2 - 2am_3 = -2a(m_1 + m_2 + m_3) = 0\)).

(iii) The centroid of the triangle formed by the co-normal points lies on the axis of the parabola [the vertices of the triangle formed by the co-normal points are \((am_1^2, -2am_1)\), \((am_2^2, -2am_2)\) and \((am_3^2, -2am_3)\). Thus, \(y\)-coordinate of the centroid becomes \(
-2a(m_1 + m_2 + m_3)
\)
\(\frac{3}{-2a} \times 0 = 0.
\)

Hence, the centroid lies on the \(x\)-axis, i.e. axis of the parabola.]
(iv) If three normals drawn to any parabola \( y^2 = 4ax \) from a given point \( (h, k) \) be real, then \( h > 2a \).

**CHORD OF CONTACT**

The equation of chord of contact of tangents drawn from a point \( P (x_1, y_1) \) to the parabola \( y^2 = 4ax \) is \( T = 0 \) where \( T = yy_1 - 2a (x + x_1) \).

**CHORD WITH A GIVEN MID POINT**

The equation of the chord of the parabola \( y^2 = 4ax \) with \( P (x_1, y_1) \) as its middle point is given by \( T = S_1 \) where \( T = yy_1 - 2a (x + x_1) \) and \( S_1 = y_1^2 - 4ax \).
at \( L, x = a \),
\[
\frac{a \cos \theta}{a} + \frac{y \sin \theta}{b} = 1
\]

\[\Rightarrow y = \frac{b}{\sin \theta} (1 - \cos \theta)\]
\[\Rightarrow AL = \frac{b}{\sin \theta} (1 - \cos \theta)\]

at \( x = -a \), \( y = \frac{b}{\sin \theta} (1 + \cos \theta)\)
\[\Rightarrow A'M = \frac{b}{\sin \theta} (1 + \cos \theta)\]

thus \( AL. A'M = \frac{b^2}{\sin^2 \theta} (1 - \cos^2 \theta) = b^2.\)

Question

If \( a, b, c \) are in A.P., \( a, x, b \) are in G.P. and \( b, y, c \) are in G.P., the point \((x, y)\) lies on
(a) a straight line
(b) a circle
(c) an ellipse
(d) a hyperbola

Ans. (b)

Solution We have \( 2b = a + c, x^2 = ab, y^2 = bc \) so that \( x^2 + y^2 = b(a + c) = 2b^2 \) which is a circle.
Question

The second degree equation \( x^2 + 3xy + 2y^2 + 3x + 5y + 2 = 0 \) represents

a) parabola  
b) ellipse  
c) hyperbola  
d) pair of straight lines

Solution

Ans (d)

Here \( a=1 \), \( h = \frac{3}{2} \), \( b=2 \), \( g = \frac{3}{2} \), \( f = \frac{5}{2} \), \( c=2 \)

thus \( abc + 2fgh - af^2 - bg^2 - ch^2 \)

\[
= 1 \cdot (2) \cdot (2) + 2 \left( \frac{5}{2} \right) \left( \frac{3}{2} \right) \left( \frac{3}{2} \right)
\]

\[
-1 \left( \frac{5}{2} \right)^2 - 2 \left( \frac{3}{2} \right)^2 - 2 \left( \frac{3}{2} \right)^2 = 0
\]

thus the second degree equation represents pair of straight lines.
Question

Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola \( \frac{x^2}{16} - \frac{y^2}{9} = 1 \).

Answer

The given equation is \( \frac{x^2}{16} - \frac{y^2}{9} = 1 \) or \( \frac{x^2}{4^2} - \frac{y^2}{3^2} = 1 \).

On comparing this equation with the standard equation of hyperbola i.e., \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \), we obtain \( a = 4 \) and \( b = 3 \).

We know that \( a^2 + b^2 = c^2 \).

\[ c^2 = 4^2 + 3^2 = 25 \]

\[ \Rightarrow c = 5 \]

Therefore,

The coordinates of the foci are \((\pm 5, 0)\).

The coordinates of the vertices are \((\pm 4, 0)\).

Eccentricity, \( e = \frac{c}{a} = \frac{5}{4} \)

Length of latus rectum \( = \frac{2b^2}{a} = \frac{2 \times 9}{4} = \frac{9}{2} \)
Question

Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola \( \frac{y^2}{9} - \frac{x^2}{27} = 1 \).

Answer

The given equation is

\[ \frac{y^2}{9} - \frac{x^2}{27} = 1 \text{ or } \frac{y^2}{3^2} - \frac{x^2}{(3\sqrt{3})^2} = 1. \]

On comparing this equation with the standard equation of hyperbola i.e., \( \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \), we obtain \( a = 3 \) and \( b = 3\sqrt{3} \).

We know that \( a^2 + b^2 = c^2 \).

\[ \therefore c^2 = 3^2 + (3\sqrt{3})^2 = 9 + 27 = 36 \]

\[ \Rightarrow c = 6 \]

Therefore,

The coordinates of the foci are \((0, \pm 6)\).

The coordinates of the vertices are \((0, \pm 3)\).

Eccentricity, \( e = \frac{c}{a} = \frac{6}{3} = 2 \)

Length of latus rectum

\[ = \frac{2b^2}{a} = \frac{2 \times 27}{3} = 18 \]
Question

Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $9y^2 - 4x^2 = 36$

Answer

The given equation is $9y^2 - 4x^2 = 36$.

It can be written as

$9y^2 - 4x^2 = 36$

Or, $\frac{y^2}{4} - \frac{x^2}{9} = 1$

Or, $\frac{y^2}{2^2} - \frac{x^2}{3^2} = 1$  ...(1)

On comparing equation (1) with the standard equation of hyperbola i.e., $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we obtain $a = 2$ and $b = 3$.

We know that $a^2 + b^2 = c^2$.

$\therefore c^2 = 4 + 9 = 13$

$\Rightarrow c = \sqrt{13}$

Therefore,

The coordinates of the foci are $\left(0, \pm \sqrt{13}\right)$.

The coordinates of the vertices are $\left(0, \pm 2\right)$

Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{13}}{2}$

Length of latus rectum $\frac{2b^2}{a} = \frac{2 \times 9}{2} = 9$
Question

Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola \(16x^2 - 9y^2 = 576\).

Answer

The given equation is \(16x^2 - 9y^2 = 576\).

It can be written as

\[
\frac{x^2}{36} - \frac{y^2}{64} = 1
\]

\[
\Rightarrow \frac{x^2}{6^2} - \frac{y^2}{8^2} = 1
\]

...(1)

On comparing equation (1) with the standard equation of hyperbola i.e., \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\), we obtain \(a = 6\) and \(b = 8\).

We know that \(a^2 + b^2 = c^2\).

\[
\therefore c^2 = 36 + 64 = 100
\]

\[
\Rightarrow c = 10
\]

Therefore,

The coordinates of the foci are \((\pm 10, 0)\).

The coordinates of the vertices are \((\pm 6, 0)\).

Eccentricity, \(e = \frac{c}{a} = \frac{10}{6} = \frac{5}{3}\)

Length of latus rectum

\[
\frac{2b^2}{a} = \frac{2 \times 64}{6} = \frac{64}{3}
\]
Question

Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $5y^2 - 9x^2 = 36$.

Answer

The given equation is $5y^2 - 9x^2 = 36$.

\[
\Rightarrow \frac{y^2}{36} - \frac{x^2}{4} = 1
\]

\[
\Rightarrow \frac{y^2}{6^2} - \frac{x^2}{2^2} = 1 \quad \text{...(1)}
\]

On comparing equation (1) with the standard equation of hyperbola \( \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \), we obtain \( a = \frac{6}{\sqrt{5}} \) and \( b = 2 \).

We know that \( a^2 + b^2 = c^2 \).

\[
\therefore c^2 = \frac{36}{5} + 4 = \frac{56}{5}
\]

\[
\Rightarrow c = \sqrt{\frac{56}{5}} = \frac{2\sqrt{14}}{\sqrt{5}}
\]
Therefore, the coordinates of the foci are \( \left( 0, \pm \frac{2\sqrt{14}}{\sqrt{5}} \right) \).

The coordinates of the vertices are \( \left( 0, \pm \frac{6}{\sqrt{5}} \right) \).

Eccentricity, \( e = \frac{c}{a} = \frac{\left( \frac{2\sqrt{14}}{\sqrt{5}} \right)}{\left( \frac{6}{\sqrt{5}} \right)} = \frac{\sqrt{14}}{3} \)

\[
\frac{2h^2}{a} = \frac{2 \times 4}{\left( \frac{6}{\sqrt{5}} \right)} = \frac{4\sqrt{5}}{3}
\]

Length of latus rectum
Question

Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola \(49y^2 - 16x^2 = 784\)

Answer

The given equation is \(49y^2 - 16x^2 = 784\).

It can be written as

\[9y^2 - 4x^2 = 784\]

\[\frac{y^2}{49} - \frac{x^2}{16} = 1\]

\[\text{Or, } \frac{y^2}{4^2} - \frac{x^2}{7^2} = 1\] \quad ...(1)

On comparing equation (1) with the standard equation of hyperbola i.e., \(\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1\), we obtain \(a = 4\) and \(b = 7\).

We know that \(a^2 + b^2 = c^2\).

\[\therefore c^2 = 16 + 49 = 65\]

\[\Rightarrow c = \sqrt{65}\]

Therefore,

The coordinates of the foci are \((0, \pm \sqrt{65})\).

The coordinates of the vertices are \((0, \pm 4)\).

Eccentricity, \(e = \frac{c}{a} = \frac{\sqrt{65}}{4}\)

Length of latus rectum \(= \frac{2b^2}{a} = \frac{2 \times 49}{4} = \frac{49}{2}\)
Question

Find the equation of the hyperbola satisfying the given conditions: Vertices $(±2, 0)$, foci $(±3, 0)$.

Answer

Vertices $(±2, 0)$, foci $(±3, 0)$

Here, the vertices are on the $x$-axis.

Therefore, the equation of the hyperbola is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Since the vertices are $(±2, 0)$, $a = 2$.

Since the foci are $(±3, 0)$, $c = 3$.

We know that $a^2 + b^2 = c^2$.

$\therefore 2^2 + b^2 = 3^2$

$b^2 = 9 - 4 = 5$

Thus, the equation of the hyperbola is $\frac{x^2}{4} - \frac{y^2}{5} = 1$.

Question

Find the equation of the hyperbola satisfying the given conditions: Vertices $(0, ±5)$, foci $(0, ±8)$.

Answer

Vertices $(0, ±5)$, foci $(0, ±8)$

Here, the vertices are on the $y$-axis.

Therefore, the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Since the vertices are $(0, ±5)$, $a = 5$.

Since the foci are $(0, ±8)$, $c = 8$.

We know that $a^2 + b^2 = c^2$.

$\therefore 5^2 + b^2 = 8^2$

$b^2 = 64 - 25 = 39$

Thus, the equation of the hyperbola is $\frac{y^2}{25} - \frac{x^2}{39} = 1$. 
Question

Find the equation of the hyperbola satisfying the given conditions: Vertices (0, ±3), foci (0, ±5).

Answer

Vertices (0, ±3), foci (0, ±5).

Here, the vertices are on the y-axis.

Therefore, the equation of the hyperbola is of the form \( \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \).

Since the vertices are (0, ±3), \( a = 3 \).

Since the foci are (0, ±5), \( c = 5 \).

We know that \( a^2 + b^2 = c^2 \).

\[ 3^2 + b^2 = 5^2 \]

\[ b^2 = 25 - 9 = 16 \]

Thus, the equation of the hyperbola is \( \frac{y^2}{9} - \frac{x^2}{16} = 1 \).

Question

Find the equation of the hyperbola satisfying the given conditions: Foci (±5, 0), the transverse axis is of length 8.

Answer

Foci (±5, 0), the transverse axis is of length 8.

Here, the foci are on the x-axis.

Therefore, the equation of the hyperbola is of the form \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \).

Since the foci are (±5, 0), \( c = 5 \).

Since the length of the transverse axis is 8, \( 2a = 8 \Rightarrow a = 4 \).

We know that \( a^2 + b^2 = c^2 \).

\[ 4^2 + b^2 = 5^2 \]

\[ b^2 = 25 - 16 = 9 \]

Thus, the equation of the hyperbola is \( \frac{x^2}{16} - \frac{y^2}{9} = 1 \).
Question

Find the equation of the hyperbola satisfying the given conditions: Foci $(0, \pm13)$, the conjugate axis is of length 24.

Answer

Foci $(0, \pm13)$, the conjugate axis is of length 24.

Here, the foci are on the $y$-axis.

Therefore, the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Since the foci are $(0, \pm13)$, $c = 13$.

Since the length of the conjugate axis is 24, $2b = 24 \Rightarrow b = 12$.

We know that $a^2 + b^2 = c^2$.

$\therefore a^2 + 12^2 = 13^2$

$\Rightarrow a^2 = 169 - 144 = 25$

Thus, the equation of the hyperbola is $\frac{y^2}{25} - \frac{x^2}{144} = 1$.

Question

Find the equation of the hyperbola satisfying the given conditions: Foci $(\pm3\sqrt{5}, 0)$, the latus rectum is of length 8.

Answer

Foci $(\pm3\sqrt{5}, 0)$, the latus rectum is of length 8.

Here, the foci are on the $x$-axis.

Therefore, the equation of the hyperbola is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Since the foci are $(\pm3\sqrt{5}, 0)$, $c = 3\sqrt{5}$.

Length of latus rectum = 8

$\Rightarrow \frac{2b^2}{a} = 8$

$\Rightarrow b^2 = 4a$
We know that $a^2 + b^2 = c^2$.
\[\Rightarrow a^2 + 4a = 45\]
\[\Rightarrow a^2 + 4a - 45 = 0\]
\[\Rightarrow a^2 + 9a - 5a - 45 = 0\]
\[\Rightarrow (a + 9)(a - 5) = 0\]
\[\Rightarrow a = -9, 5\]

Since $a$ is non-negative, $a = 5$.
\[\Rightarrow b^2 = 4a = 4 \times 5 = 20\]

Thus, the equation of the hyperbola is $rac{x^2}{25} - rac{y^2}{20} = 1$.

**Question**

Find the equation of the hyperbola satisfying the given conditions: Foci $(\pm 4, 0)$, the latus rectum is of length 12.

**Answer**

Foci $(\pm 4, 0)$, the latus rectum is of length 12.

Here, the foci are on the $x$-axis.

Therefore, the equation of the hyperbola is of the form $rac{x^2}{a^2} - rac{y^2}{b^2} = 1$.

Since the foci are $(\pm 4, 0)$, $c = 4$.

Length of latus rectum $= 12$

\[\Rightarrow \frac{2b^2}{a} = 12\]
\[\Rightarrow b^2 = 6a\]

We know that $a^2 + b^2 = c^2$.
\[\Rightarrow a^2 + 6a = 16\]
\[\Rightarrow a^2 + 6a - 16 = 0\]
\[\Rightarrow a^2 + 8a - 2a - 16 = 0\]
\[\Rightarrow (a + 8)(a - 2) = 0\]
\[\Rightarrow a = -8, 2\]

Since $a$ is non-negative, $a = 2$.
\[\Rightarrow b^2 = 6a = 6 \times 2 = 12\]
Thus, the equation of the hyperbola is \( \frac{x^2}{4} - \frac{y^2}{12} = 1 \).

Question

Find the equation of the hyperbola satisfying the given conditions: Vertices \((\pm 7, 0)\),
\[ e = \frac{4}{3} \]

Answer

Vertices \((\pm 7, 0)\),
\[ e = \frac{4}{3} \]
Here, the vertices are on the x-axis.

Therefore, the equation of the hyperbola is of the form \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \).

Since the vertices are \((\pm 7, 0)\), \(a = 7\).

It is given that \(e = \frac{4}{3}\)

\[ \therefore \frac{c}{a} = \frac{4}{3} \]
\[ \Rightarrow \frac{c}{7} = \frac{4}{3} \]
\[ \Rightarrow c = \frac{28}{3} \]

We know that \(a^2 + b^2 = c^2\).

\[ \therefore 7^2 + b^2 = \left(\frac{28}{3}\right)^2 \]
\[ \Rightarrow b^2 = \frac{49}{9} \cdot 49 \]
\[ \Rightarrow b^2 = \frac{784 - 441}{9} = \frac{343}{9} \]

Thus, the equation of the hyperbola is
\[ \frac{x^2}{49} - \frac{9y^2}{343} = 1 \]
Question

Find the equation of the hyperbola satisfying the given conditions: Foci $\left(0, \pm \sqrt{10}\right)$, passing through $(2, 3)$

Answer

Foci $\left(0, \pm \sqrt{10}\right)$, passing through $(2, 3)$

Here, the foci are on the y-axis.

Therefore, the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Since the foci are $\left(0, \pm \sqrt{10}\right)$, $c = \sqrt{10}$.

We know that $a^2 + b^2 = c^2$.

$\therefore a^2 + b^2 = 10$

$\Rightarrow b^2 = 10 - a^2 \quad \ldots (1)$

Since the hyperbola passes through point $(2, 3)$,

$\frac{9}{a^2} - \frac{4}{b^2} = 1 \quad \ldots (2)$

From equations (1) and (2), we obtain

$\frac{9}{a^2} - \frac{4}{10 - a^2} = 1$

$\Rightarrow 9(10 - a^2) - 4a^2 = a^2(10 - a^2)$
\[ 90 - 9a^2 - 4a^2 = 10a^2 - a^4 \]
\[ \Rightarrow a^4 - 23a^2 + 90 = 0 \]
\[ \Rightarrow 4a^2 - 18a^2 - 5a^2 + 90 = 0 \]
\[ \Rightarrow a^2 (a^2 - 18) - 5 (a^2 - 18) = 0 \]
\[ \Rightarrow (a^2 - 18)(a^2 - 5) = 0 \]
\[ \Rightarrow a^2 = 18 \text{ or } 5 \]

In hyperbola, \( c > a \), i.e., \( c^2 > a^2 \)
\[ \Rightarrow a^2 = 5 \]
\[ \Rightarrow b^2 = 10 - a^2 = 10 - 5 = 5 \]

Thus, the equation of the hyperbola is
\[ \frac{y^2}{5} - \frac{x^2}{5} = 1 \]

**Question**

Let \( S(-1,1) \) be the focus and \( P(x,y) \) be a point on the hyperbola. Draw \( PM \) perpendicular from \( P \) on the directrix. Then, by definition,

\[ SP = ePM \]
\[ \Rightarrow SP^2 = e^2PM^2 \]
\[ \Rightarrow (x + 1)^2 + (y - 1)^2 = (3)^2 \left[ \frac{x - y + 3}{\sqrt{x^2 + (-1)^2}} \right]^2 \]
\[ = x^2 + 1 + 2x + y^2 + 1 - 2y = \frac{9[x - y + 3]^2}{2} \]
\[ \Rightarrow 2 \left[ x^2 + y^2 + 2x - 2y + 2 \right] = 9[x - y + 3]^2 \]
\[ = 2(x^2 + 2y^2 + 4x - 4y + 4) = 9[x - y + 3]^2 \]
\[ = 2x^2 + 2y^2 + 4x - 4y + 4 = 9(x^2 + y^2 + 9) - 2xy - 6y + 6x \]
\[ = 2x^2 + 7y^2 - 18xy + 50y - 50y + 77 = 0 \]

This is the required equation of the hyperbola.
Question

Let $S(0, 3)$ be the focus and $P(x, y)$ be a point on the hyperbola. Draw $PM$ perpendicular from $P$ on the directrix. Then, by definition

$$
\begin{align*}
\vec{sP} &= \vec{ePM} \\
\vec{sP}^2 &= e^2\vec{PM}^2 \\
\Rightarrow \quad (x - 0)^2 + (y - 3)^2 &= 2^2 \left[ \frac{x + y - 1}{\sqrt{1^2 + 1^2}} \right]^2 \\
\Rightarrow \quad x^2 + y^2 - 6y &= \frac{4(x + y - 1)^2}{2} \\
\Rightarrow \quad x^2 + y^2 - 6y + 9 &= 2(x + y - 1)^2 \\
\Rightarrow \quad x^2 + y^2 - 6y + 9 &= 2 \left[ x^2 + y^2 + (-1)^2 + 2xy + 2xy \cdot (-1) + 2 \cdot (-1) \cdot x \right] \\
\Rightarrow \quad x^2 + y^2 - 6y + 9 &= 2 \left[ x^2 + y^2 + 2 + 2xy - 2y - 2x \right] \\
\Rightarrow \quad x^2 + y^2 - 6y + 9 &= 2x^2 + 2y^2 + 2 + 4xy - 4y - 4x \\
\Rightarrow \quad 2x^2 - x^2 + 2y^2 + 4xy - 4x - 4y + 5y + 2 - 9 &= 0 \\
\Rightarrow \quad x^2 + y^2 + 4xy - 4x + 2y - 7 &= 0
\end{align*}
$$

This is the required equation of the hyperbola.

Let $S(1, 1)$ be the focus and $P(x, y)$ be a point on the hyperbola. Draw $PM$ perpendicular from $P$ on the directrix. Then, by definition

$$
\begin{align*}
\vec{sP} &= \vec{ePM} \\
\vec{sP}^2 &= e^2\vec{PM}^2 \\
\Rightarrow \quad (x - 1)^2 + (y - 1)^2 &= 2^2 \left[ \frac{3x + 4y + 8}{\sqrt{3^2 + 4^2}} \right]^2 \\
\Rightarrow \quad x^2 + 1 - 2x + y^2 + 1 - 2y &= \frac{4(3x + 4y + 8)^2}{25} \\
\Rightarrow \quad x^2 + y^2 - 2x - 2y + 2 &= \frac{4(3x + 4y + 8)^2}{25} \\
\Rightarrow \quad 25x^2 + 2\cdot 25y^2 - 50x - 50y + 50 &= 4(3x + 4y + 8)^2 \\
\Rightarrow \quad 25x^2 + 2\cdot 25y^2 - 50x - 50y + 50 &= 4 \left[ 5x^2 + 16y^2 + 6y + 24xy + 64y + 48x \right] \\
\Rightarrow \quad 25x^2 + 2\cdot 25y^2 - 50x - 50y + 50 &= 36x^2 + 64y^2 + 255 + 96xy + 256y + 192x \\
\Rightarrow \quad 36x^2 - 25x^2 + 64y^2 - 25y^2 + 96xy + 192x + 50x + 256y + 50y + 256 - 50 &= 0 \\
\Rightarrow \quad 11x^2 + 39y^2 + 96xy + 242x + 300y + 206 &= 0
\end{align*}
$$

This is the required equation of the hyperbola.
Question

Let \( S (1, 1) \) be the focus and \( P (x, y) \) be a point on the hyperbola.
Draw \( PM \) perpendicular from \( P \) on the directrix. Then, by definition

\[
sp^2 = sp \cdot PM
\]

\[
sp^2 = s^2 PM^2
\]

\[
(x - 1)^2 + (y - 1)^2 = (\sqrt{3})^2 \left[ \frac{2x + y - 1}{\sqrt{x^2 + 1}} \right]^2 [\because \theta = 2]
\]

\[
x^2 + 1 - 2x + y^2 + 1 - 2y = \frac{3(2x + y - 1)}{5}
\]

\[
5 \left[ x^2 + y^2 - 2x - 2y + 2 \right] = 3(2x + y - 1)^2
\]

\[
5x^2 + 5y^2 - 10x - 10y + 10 = 3 \left[ (2x)^2 + y^2 + (-1)^2 + 2 \cdot 2x \cdot y + 2 \cdot x \cdot y \cdot (-1) + 2 \cdot (-1) \cdot 2x \right]
\]

\[
5x^2 + 5y^2 - 10x - 10y + 10 = 3 \left[ 4x^2 + y^2 + 1 + 4xy - 2y - 4x \right]
\]

\[
5x^2 + 5y^2 - 10x - 10y + 10 = 12x^2 + 3y^2 + 3 + 12xy - 6y - 12x
\]

\[
12x^2 - 5x^2 + 3y^2 - 5y^2 + 12xy - 12x + 10x - 6y + 10y = 3 - 10 = 0
\]

\[
7x^2 - 2y^2 + 12xy - 2x + 4y - 7 = 0
\]

This is the required equation of the hyperbola.

Question

Let \( S (2, -1) \) be the focus and \( P (x, y) \) be a point on the hyperbola.
Draw \( PM \) perpendicular from \( P \) on the directrix. Then, by definition

\[
sp^2 = sp \cdot PM
\]

\[
sp^2 = s^2 PM^2
\]

\[
(x - 2)^2 + (y + 1)^2 = 2^2 \left[ \frac{2x + 3y - 1}{\sqrt{2^2 + 3^2}} \right]^2 [\because \theta = 2]
\]

\[
x^2 + 4 - 4x + y^2 + 1 + 2y = \frac{4(2x + 3y - 1)^2}{13}
\]

\[
13 \left[ x^2 + y^2 - 4x - 2y + 5 \right] = 4 \left( 2x + 3y - 1 \right)^2
\]

\[
13x^2 + 13y^2 - 52x + 26y + 65 = 4 \left[ (2x)^2 + (3y)^2 + (-1)^2 + 2 \cdot 2x \cdot 3y + 2 \cdot 3y \cdot (-1) + 2 \cdot (-1) \cdot 2x \right]
\]

\[
13x^2 + 13y^2 - 52x + 26y + 65 = 16x^2 + 36y^2 + 4 + 48xy - 24y - 16x
\]

\[
16x^2 - 13x^2 + 36y^2 - 13y^2 + 48xy - 16x + 52x - 24y - 26y + 4 - 65 = 0
\]

\[
3x^2 + 23y^2 + 48xy + 36x - 50y - 61 = 0
\]

This is the required equation of the hyperbola.
Question

Let \( S (a, 0) \) be the focus and \( P (x, y) \) be a point on the hyperbola.

Draw \( PM \) perpendicular from \( P \) on the directrix. Then, by definition

\[
sp = ePM
\]

\[
s^2 = e^2PM^2
\]

\[
(x - a)^2 + (y - 0)^2 = \left( \frac{4}{3} \right)^2 \frac{2x - y + a}{\sqrt{2^2 + (-1)^2}}
\]

\[
\therefore e = \frac{4}{3}
\]

\[
x^2 + a^2 - 2ax + y^2 = \frac{16}{9} \times \frac{[2x - y + a]^2}{5}
\]

\[
45x^2 + 2y^2 - 2ax + a^2 = 16[2x - y + a]^2
\]

\[
45x^2 + 2y^2 - 90ax + 45a^2 = 15(2x^2 + (-y)^2 + 2 \times 2x \times (-y) + 2 \times (-y) \times x + 2 \times a \times x + 2 \times a \times x)
\]

\[
45x^2 + 2y^2 - 90ax + 45a^2 = 15[4x^2 + y^2 + a^2 - 2xy - 32ay + 64ax]
\]

\[
45x^2 + 2y^2 - 90ax + 45a^2 = 64x^2 + 16y^2 + 16a^2 - 64xy - 32ay - 64ax
\]

\[
64x^2 - 45x^2 + 16y^2 - 45y^2 - 64xy + 64ax + 90ax - 32ay + 16a^2 - 45a^2 = 0
\]

\[
19x^2 - 29y^2 - 64xy + 154ax - 64ay - 29a^2 = 0
\]

This is the required equation of the hyperbola.

Question

Let \( S (2, 2) \) be the focus and \( P (x, y) \) be a point on the hyperbola.

Draw \( PM \) perpendicular from \( P \) on the directrix. Then, by definition

\[
sp = ePM
\]

\[
s^2 = e^2PM^2
\]

\[
(x - 2)^2 + (y - 2)^2 = 2 \times \frac{x + y - 9}{\sqrt{1^2 + 1^2}}
\]

\[
\therefore e = \frac{4}{3}
\]

\[
x^2 + 4 - 4x + y^2 + 4 - 4y = \frac{4(x + y - 9)}{2}
\]

\[
x^2 + y^2 - 4x - 4y + 8 = 2(x + y - 9)
\]

\[
x^2 + y^2 - 4x - 4y + 8 = 2(x^2 + y^2 + (9))^2 + 2 \times x \times y + 2 \times y \times x^2 \times y \times (-9) + 2 \times x(-9) \times x
\]

\[
x^2 + y^2 - 4x - 4y + 8 = 2(x^2 + y^2 + 81 + 2xy - 18y + 18x)
\]

\[
x^2 + y^2 - 4x - 4y + 8 = 2(x^2 + y^2 + 162 + 4xy - 36y - 36x)
\]

\[
2x^2 - x^2 + 2y^2 - y^2 + 4xy - 36x + 4x - 36y + 4y + 162 - 8 = 0
\]

\[
x^2 + y^2 + 4xy - 32x - 32y + 154 = 0
\]

This is the required equation of the hyperbola.
Question

We have,

\[9x^2 - 16y^2 = 144\]

\[\Rightarrow \frac{9x^2}{144} - \frac{16y^2}{144} = 1\]

\[\Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1\]

This is of the form \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\), where \(a^2 = 16\) and \(b^2 = 9\).

Eccentricity: The eccentricity \(e\) is given by

\[e = \sqrt{1 + \frac{b^2}{a^2}}\]

\[= \sqrt{1 + \frac{9}{16}}\]

\[= \frac{\sqrt{25}}{\sqrt{16}}\]

\[= \frac{5}{4}\]

Focus: The coordinates of the foci are \((\pm ae, 0)\) i.e., \((\pm 5, 0)\).

Equations of the directrices: The equations of the directrices are

\[x = \pm \frac{a}{e}\] i.e., \(x = \pm \frac{16}{5}\)

\[\Rightarrow 5x = \pm 16\]

\[\Rightarrow 5x \mp 16 = 0\]
Length of latus-rectum: The length of the latus-rectum

\[ \frac{2b^2}{a} = \frac{2 \times 9}{4} = \frac{9}{2} \]

We have,

\[ 16x^2 - 9y^2 = -144 \]

\[ \Rightarrow \frac{16x^2}{144} - \frac{9y^2}{144} = -1 \]

\[ \Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = -1 \]

This is of the form \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \), where \( a^2 = 9 \) and \( b^2 = 16 \)

\[ a = 3 \text{ and } b = 4 \]

Eccentricity: The eccentricity \( e \) is given by

\[ e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4} \]

Foci: The coordinates of the foci are \((0, \pm be)\).

\[ (0, \pm be) = \left(0, \pm 4 \times \frac{5}{4}\right) = (0, \pm 5) \]

Equations of the directrices: The equations of the directrices are

\[ y = \pm \frac{eb}{e} \]

\[ \Rightarrow y = \pm \frac{4 \times 5}{5} = \pm \frac{16}{5} \]

\[ = \frac{5y + 16}{5} = 0 \]

Latus-rectum: The length of the latus-rectum

\[ \frac{2a^2}{b} = \frac{2 \times 9}{4} = \frac{9}{2} \]
Question

We have,

\[ 4x^2 - 3y^2 = 36 \]

\[ \Rightarrow \quad \frac{4x^2}{36} - \frac{3y^2}{36} = 1 \]

\[ \Rightarrow \quad \frac{x^2}{9} - \frac{y^2}{12} = 1 \]

This is of the form \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \), where \( a^2 = 9 \) and \( b^2 = 12 \)

\[ \therefore a = 3 \quad \text{and} \quad b = \sqrt{12} = 2\sqrt{3} \]

Eccentricity: The eccentricity \( e \) is given by

\[ e = \sqrt{1 + \frac{b^2}{a^2}} \]

\[ = \sqrt{1 + \frac{12}{9}} \]

\[ = \sqrt{1 + \frac{4}{3}} \]

\[ = \sqrt{\frac{7}{3}} \]
Foci: The coordinates of the foci are \((\pm ae, 0)\).

\[
\pm ae = \pm 3 \times \sqrt{\frac{7}{3}}
\]

\[
= \pm 3 \times \frac{\sqrt{7}}{\sqrt{3}}
\]

\[
= \pm \sqrt{21}
\]

\[
\therefore \ (\pm ae, 0) = \left(\pm \sqrt{21}, 0\right)
\]

the coordinates of the foci are \(\left(\pm \sqrt{21}, 0\right)\)

Equations of the directrices: The equations of the directrices are

\[
x = \frac{\pm a}{e}
\]

\[
x = \pm 3 \times \frac{1}{\sqrt{\frac{7}{3}}}
\]

\[
= \pm 3 \times \frac{\sqrt{3}}{\sqrt{7}}
\]

\[
\Rightarrow \sqrt{7}x \mp 3\sqrt{3} = 0
\]

The equations of the directrices are \(\sqrt{7}x \mp 3\sqrt{3} = 0\)
Latus-rectum: The length of the latus-rectum
\[ \frac{2b^2}{a} = \frac{2 \times 12}{3} = 8 \]

We have,
\[ 3x^2 - y^2 = 4 \]
\[ \Rightarrow \frac{3x^2}{4} - \frac{y^2}{4} = 1 \]
\[ \Rightarrow \frac{x^2}{4} - \frac{y^2}{4} = 1 \]
\[ \Rightarrow \frac{x^2}{2^2} - \frac{y^2}{(\sqrt{3})^2} = 1 \]

This is of the form \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \), where \( a = \frac{2}{\sqrt{3}} \) and \( b = 2 \)

Eccentricity: The eccentricity \( e \) is given by
\[ e = \sqrt{1 + \frac{b^2}{a^2}} \]
\[ = \sqrt{1 + \frac{4}{4}} \]
\[ = \sqrt{1 + 3} \]
\[ = \sqrt{4} \]
\[ = 2 \]

Foci: The coordinates of the foci are \( (\pm ae, 0) \)
\[ \therefore \quad \pm ae = \pm \frac{2}{\sqrt{3}} \times 2 = \pm \frac{4}{\sqrt{3}} \]

The coordinates of the foci are \( \left( \pm \frac{4}{\sqrt{3}}, 0 \right) \)

Equations of the directrices: The equations of the directrices are
\[ x = \pm \frac{a}{e} \]
\[ = \pm \frac{2}{\sqrt{3}} \times 2 \]
\[ = \pm \frac{4}{\sqrt{3}} \]
\[ \Rightarrow \quad \sqrt{3}x \pm 1 = 0 \]

Latus-rectum: The length of the latus-rectum is \( \frac{2b^2}{a} \)
\[ \therefore \quad \frac{2b^2}{a} = 2 \times \frac{4}{2} \]
\[ = 4 \sqrt{3} \]
Question

We have,

\[\frac{25x^2 - 36y^2}{225} = 1\]

\[\Rightarrow \frac{x^2}{9} - \frac{y^2}{25} = 1\]

\[\Rightarrow \frac{x^2}{3^2} - \frac{y^2}{5^2} = 1\]

This is of the form \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\), where \(a = 3\) and \(b = \frac{5}{2}\).

Length of the transverse axis: The length of the transverse axis

\[= 2a\]

\[= 2 \times 3 = 6\]

Length of the conjugate axis: The length of the conjugate axis is

\[2b = 2 \times \frac{5}{2} = 5\]

Eccentricity: The eccentricity \(e\) is given by
\[ e = \sqrt{1 + \frac{b^2}{a^2}} \]
\[ = \sqrt{1 + \frac{25}{36}} \]
\[ = \sqrt{\frac{61}{36}} \]
\[ = \frac{\sqrt{61}}{6} \]

Length of LR = \[ \frac{2b^2}{a} = \frac{25}{6} \]
Foci \((\pm \frac{\sqrt{61}}{2}, 0)\)
Question

We have,

\[15x^2 - 9y^2 + 32x + 36y - 164 = 0\]
\[\Rightarrow 15x^2 + 32x - 9y^2 + 36y = 14 = 0\]
\[\Rightarrow 15\left(x^2 + 2x\right) - 9\left(y^2 + 4y\right) = 164 = 0\]
\[\Rightarrow 15\left(x + 1\right)^2 - 1 - 9\left(y + 2\right)^2 - 4 = 164 = 0\]
\[\Rightarrow 15\left(x + 1\right)^2 - 10 - 9\left(y + 2\right)^2 = 164 = 0\]
\[\Rightarrow 15\left(x + 1\right)^2 - 9\left(y + 2\right)^2 + 36 - 164 = 0\]
\[\Rightarrow 15\left(x + 1\right)^2 - 9\left(y + 2\right)^2 + 20 = 0\]
\[\Rightarrow 15\left(x + 1\right)^2 - 9\left(y + 2\right)^2 = 144 = 0\]
\[\Rightarrow \frac{15\left(x + 1\right)^2}{144} - \frac{9\left(y + 2\right)^2}{144} = 1\]
\[\Rightarrow \frac{(x + 1)^2}{9} - \frac{(y + 2)^2}{16} = 1\] \hspace{1cm} \text{---(i)}

Shifting the origin at \((-1, 2)\) without rotating the coordinate axes and denoting the new coordinates with respect to the new axes by \(X\) and \(Y\),

We have,

\[x = X - 1 \text{ and } y = Y + 2\] \hspace{1cm} \text{---(ii)}

This is of the form \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\), where \(a^2 = 9\) and \(b^2 = 16\). So,
We have,
Centre: The coordinates of the centre w.r.t the new axes are \(X = 0, Y = 0\)

\[x = -1 \text{ and } y = 2 \quad \text{[Using equation (ii)]}\]

So, the coordinates of the centre w.r.t the old axes are \((-1, 2)\).

Eccentricity: The eccentricity \(e\) is given by

\[
e = \sqrt{1 + \frac{b^2}{a^2}}
\]

\[= \sqrt{1 + \frac{16}{9}}
\]

\[= \frac{5}{3}
\]

Foci: The coordinates of the foci with respect to the new axes are given by \((X = \pm ae, Y = 0)\)

i.e., \((X = \pm 5, Y = 0)\).

Putting \(X = \pm 5\) and \(Y = 0\) in equation (ii), we get
\[ x = \pm 5 - 1 \text{ and } y = 0 + 2 \]
\[ \Rightarrow x = 4, -6 \text{ and } y = 2 \]

Equation of the directrix: The equations of the directrix are

\[ x' = \pm \frac{a}{e} \]
\[ = \pm \frac{3}{5} \]
\[ x' = \pm \frac{9}{5} \]

Putting \( x' = \pm \frac{9}{5} \) in equation (ii), we get

\[ x = \pm \frac{9}{5} - 1 \]
\[ \Rightarrow x = \frac{4}{5} - \frac{5}{5} \]
\[ \Rightarrow x = \frac{4}{5} \text{ and } x = \frac{-14}{5} \]
\[ \Rightarrow 5x - 4 = 0 \text{ and } 5x + 14 = 0 \]

So, the equations of the directrices w.r.t the old axes are
\[ 5x - 4 = 0 \text{ and } 5x + 14 = 0. \]

We have,
\[ x^2 - y^2 + 4x = 0 \]
\[ \Rightarrow x^2 + 4x - y^2 = 0 \]
\[ \Rightarrow x^2 + 4x + 4 - 4 - y^2 = 0 \]
\[ \Rightarrow (x + 2)^2 - y^2 = 4 \]
\[ \Rightarrow \frac{(x + 2)^2}{4} - \frac{y^2}{4} = 1 \quad \text{---(i)} \]

Shifting the origin at \((-2, 0)\) without rotating the axes and denoting the new coordinates w.r.t these axes by \(X\) and \(Y\),

We have,
\[ X = x - 2 \text{ and } Y = y \quad \text{---(ii)} \]

Using these relations, equation (i) reduces to
\[ \frac{X^2}{4} - \frac{Y^2}{4} = 1 \quad \text{---(ii)} \]

This is of the form \( \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1 \), where \(a^2 = 4\) and \(b^2 = 4\). So,

We have,
Centre: The coordinates of the centre w.r.t the new axes are \((X = 0, Y = 0)\)

Putting \(X = 0\) and \(Y = 0\) in equation (ii), we get
\[ x = -2 \text{ and } y = 0. \]

So, the coordinates of the centre w.r.t the old axes are \((-2, 0)\).
To recall standard integrals

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$\int f(x) , dx$</th>
<th>$f(x)$</th>
<th>$\int f(x) , dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$\frac{x^{n+1}}{n+1}$ $(n \neq -1)$</td>
<td>$[g(x)]^n \cdot g'(x)$</td>
<td>$\frac{[g(x)]^{n+1}}{n+1}$ $(n \neq -1)$</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln</td>
<td>x</td>
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</tr>
<tr>
<td>$e^x$</td>
<td>$e^x$</td>
<td>$e^x$</td>
<td>$a^x \cdot \frac{a}{\ln a}$ $(a &gt; 0)$</td>
</tr>
<tr>
<td>$\sin x$</td>
<td>$-\cos x$</td>
<td>$\sinh x$</td>
<td>$\cosh x$</td>
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<tr>
<td>$\cos x$</td>
<td>$\sin x$</td>
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<td>$\sec^2 x$</td>
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<td>$\sinh^2 x$</td>
<td>$\frac{\sinh 2x - x}{2}$</td>
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<td>$\cos^2 x$</td>
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<td>$\cosh^2 x$</td>
<td>$\frac{\sinh 2x + x}{2}$</td>
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<tbody>
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<td>$\frac{1}{a} \tan^{-1} \frac{x}{a}$ $(a &gt; 0)$</td>
<td>$\frac{1}{a^2 - x^2}$</td>
<td>$\frac{1}{a^2 - x^2}$ $(a &gt; 0)$</td>
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<tr>
<td>$\frac{1}{\sqrt{a^2 - x^2}}$</td>
<td>$\sin^{-1} \frac{x}{a}$ $(a &gt; 0)$</td>
<td>$\frac{1}{\sqrt{a^2 + x^2}}$</td>
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<tr>
<td>$\sqrt{a^2 - x^2}$</td>
<td>$\frac{a^2}{2} \left[ \sin^{-1} \left( \frac{x}{a} \right) + \frac{x \sqrt{a^2 - x^2}}{a^2} \right]$</td>
<td>$\sqrt{a^2 + x^2}$</td>
<td>$\frac{a^2}{2} \left[ \sinh^{-1} \left( \frac{x}{a} \right) + \frac{x \sqrt{a^2 + x^2}}{a^2} \right]$</td>
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<tr>
<td>$\frac{1}{\sqrt{x^2 - a^2}}$</td>
<td>$-\cos^{-1} \left( \frac{x}{a} \right)$</td>
<td>$\frac{1}{\sqrt{x^2 - a^2}}$</td>
<td>$\frac{\sqrt{x^2 - a^2}}{a^2}$</td>
</tr>
</tbody>
</table>
Some series Expansions -

\[
\frac{\pi}{2} = \left(\frac{2}{1}\right) \left(\frac{2}{3}\right) \left(\frac{4}{5}\right) \left(\frac{4}{7}\right) \left(\frac{6}{5}\right) \left(\frac{6}{7}\right) \left(\frac{8}{9}\right) \left(\frac{8}{7}\right) \cdots
\]

\[
\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \cdots
\]

\[
\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots
\]

\[
\pi = \sqrt{12} \left(1 - \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{5} \cdot \frac{1}{5} - \frac{1}{7} \cdot \frac{1}{7} + \cdots\right)
\]

\[
\frac{x^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

\[
\int_{0}^{\pi} \log \sin x \, dx = -\frac{\pi}{2} \log 2 = \frac{\pi}{2} \log \frac{1}{2}
\]

Solve a series problem

If \(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\) up to \(\infty\) is \(\frac{\pi^2}{6}\), then value of

\[
\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\) up to \(\infty\) is

(a) \(\frac{\pi^2}{4}\)  (b) \(\frac{\pi^2}{6}\)  (c) \(\frac{\pi^2}{8}\)  (d) \(\frac{\pi^2}{12}\)

**Ans. (c)**

**Solution** We have

\[
\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\) up to \(\infty\)
\]

\[
= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots\) up to \(\infty\)
\]

\[
= \frac{\pi^2}{6} - \frac{1}{4}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{8}
\]

\[
1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots\) up to \(\infty\) = \(\frac{\pi^2}{12}\)
\]

\[
\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\) up to \(\infty\) = \(\frac{\pi^2}{24}\)
\]
\[
\sin \frac{\sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{3!} + \frac{x^3}{5!} - \frac{x^5}{7!} + \frac{x^7}{9!} - \frac{x^9}{11!} + \ldots
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k + 1)!}
\]

\[
\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}
\]

\[
\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k + 1)!}
\]

\[
\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \quad (-1 \leq x < 1)
\]

\[
\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots \quad |x| < \frac{\pi}{2}
\]

\[
\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots + \frac{B_n x^{2n}}{(2n)!} + \ldots \quad |x| < \frac{\pi}{2}
\]

\[
\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \ldots + \frac{2(2^{2n-1}-1)B_n x^{2n-1}}{(2n)!} + \ldots \quad 0 < |x| < \pi
\]

\[
\cot x = \frac{1}{x} - \frac{x^3}{3} - \frac{2x^5}{45} - \frac{17x^7}{945} + \ldots - \frac{2^{2n} B_n x^{2n-1}}{(2n)!} + \ldots \quad 0 < |x| < \pi
\]
\[ \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \]
\[ \sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{4} + \cdots \]
\[ \log (\cos x) = -\frac{x^2}{2} - \frac{2x^4}{4} - \cdots \]
\[ \log (x + \sin x) = x - \frac{x^3}{2} + \frac{x^5}{6} - \frac{x^7}{12} + \cdots \]

\[ \sin^{-1} x = x + \frac{x^3}{2} + \frac{1\cdot 3\cdot x^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5\cdot x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots \quad |x| < 1 \]

\[ \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \]
\[ = \frac{\pi}{2} \left( x + \frac{1\cdot x^3}{2\cdot 3} + \frac{1\cdot 3\cdot x^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5\cdot x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots \right) \quad |x| < 1 \]

\[ \tan^{-1} x = \left\{ \begin{array}{ll}
\frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots & |x| < 1 \\
\pm \frac{1}{x} + \frac{1}{3\cdot x^3} - \frac{1}{5\cdot x^5} + \cdots & +1 \text{ if } x \geq 1 \\
\pm \frac{1}{x} - \frac{1}{3\cdot x^3} + \frac{1}{5\cdot x^5} + \cdots & -1 \text{ if } x \leq -1 
\end{array} \right. \]

\[ \sec^{-1} x = \cos^{-1} \left( \frac{1}{x} \right) \]
\[ = \frac{\pi}{2} \left( x + \frac{1}{2\cdot 3\cdot x^3} + \frac{1\cdot 3\cdot x^5}{2\cdot 4\cdot 5\cdot x^5} + \frac{1\cdot 3\cdot 5\cdot x^7}{2\cdot 4\cdot 6\cdot 7\cdot x^7} + \cdots \right) \quad |x| > 1 \]

\[ \csc^{-1} x = \sin^{-1} \left( \frac{1}{x} \right) \]
\[ = \frac{1}{x} + \frac{1}{2\cdot 3\cdot x^3} + \frac{1\cdot 3\cdot x^5}{2\cdot 4\cdot 5\cdot x^5} + \frac{1\cdot 3\cdot 5\cdot x^7}{2\cdot 4\cdot 6\cdot 7\cdot x^7} + \cdots \quad |x| > 1 \]

\[ \cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x \]
\[ = \left\{ \begin{array}{ll}
\frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots & |x| < 1 \\
px + \frac{1}{3\cdot x^3} + \frac{1}{5\cdot x^5} + \cdots & p = 0 \text{ if } x \geq 1 \\
px - \frac{1}{3\cdot x^3} + \frac{1}{5\cdot x^5} + \cdots & p = 1 \text{ if } x \leq -1 
\end{array} \right. \]
\[
\ln(1+x) = 1 - \frac{x}{2} + \frac{x^2}{3^2} - ... = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad (|x| < 1)
\]

\[
\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ... = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} \quad (-1 \leq x < 1)
\]

\[
\log_e(1+x) - \log_e(1-x) = 2 \left( x + \frac{x^3}{3} + \frac{x^5}{5} + ... \right) \quad (-1 < x < 1)
\]

\[
\log_e \left( 1 + \frac{1}{x} \right) = \log_e \frac{x+1}{x} = 2 \left[ \frac{1}{2x+1} + \frac{1}{3(2x+1)^3} + \frac{1}{5(2x+1)^5} + ... \right]
\]

\[
\log_e (1 + x) + \log_e (1 - x) = \log_e (1 - x^2) = -2 \left( \frac{x^2}{2} + \frac{x^4}{4} + ... \right) \quad (-1 < x < 1)
\]

\[
\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - ... = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + ...
\]
Important Results

(i) \( \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} \, dx = \frac{\pi}{4} - \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x} \, dx \)

(ii) \( \int_0^{\pi/2} \frac{\tan^n x}{1 + \tan^n x} \, dx = \frac{\pi}{4} - \int_0^{\pi/2} \frac{dx}{1 + \tan^n x} \)

(iii) \( \int_0^{\pi/2} \frac{\cot^n x}{1 + \cot^n x} \, dx = \frac{\pi}{4} - \int_0^{\pi/2} \frac{\cot^n x}{1 + \cot^n x} \, dx \)

(iv) \( \int_0^{\pi/2} \frac{x^n}{\sec^n x + \cosec^n x} \, dx = \frac{\pi}{4} - \int_0^{\pi/2} \frac{x^n}{\sec^n x + \cosec^n x} \, dx \) where, \( n \in \mathbb{R} \)

(ii) \( \int_0^{\pi/2} \frac{a^{\sin^n x}}{a^{\sin^n x} + a^{\cos^n x}} \, dx = \int_0^{\pi/2} \frac{a^{\cos^n x}}{a^{\sin^n x} + a^{\cos^n x}} \, dx = \frac{\pi}{4} \)

(iii) \( \int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2} \log 2 \)

\( \int_0^{\pi/2} \log x \, dx = \int_0^{\pi/2} \log \cot x \, dx = 0 \)

\( \int_0^{\pi/2} \log \sec x \, dx = \int_0^{\pi/2} \log \cosec x \, dx = \frac{\pi}{2} \log 2 \)

(iv) \( \int_0^{\pi/2} e^{ax} \sin bx \, dx = \frac{b}{a^2 + b^2} \)

\( \int_0^{\pi/2} e^{ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \)

\( \int_0^{\pi/2} e^{ax} x^n \, dx = \frac{n!}{a^n + 1} \)
\[ \int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left( x + \sqrt{x^2 - a^2} \right) + C \]
\[ \int \frac{dx}{\sqrt{x^2 + a^2}} = \ln \left( x + \sqrt{x^2 + a^2} \right) + C \]
\[ \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left( \frac{x - a}{x + a} \right) + C \]
\[ \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left( \frac{a + x}{a - x} \right) + C \]
\[ \int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left( \frac{x}{a} \right) + C \]
\[ \int \sqrt{a^2 + x^2} \, dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \sinh^{-1} \left( \frac{x}{a} \right) + C \]
\[ \int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cosh^{-1} \left( \frac{x}{a} \right) + C \]
Good Luck to you for your Preparations, References, and Exams

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